Dynamic Mechanism Design with Capacity Constraint*

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Abstract

We study a project assignment problem, where a principal needs to assign multiple projects to an agent. The agent is privately informed about her cost, which could be high or low. The agent's type evolves stochastically over time. In the optimal mechanism, a low-cost agent is always allocated with projects. A high-cost agent is never allocated with a project if it is efficient (surplus maximizing) to do so; otherwise, optimal allocations to a high-cost agent are further distorted below the first-best level. We fully characterize the optimal mechanism via a sequence of deadlines. We show that the presence of the capacity constraint could delay the assignment of the projects and reduce the principal's payoff.

Keywords: Dynamic Mechanism Design; Capacity Constraint; Deadline; Efficiency

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1 Introduction

Consider a principal/project manager (he) working for an IT firm that needs to complete a large project. The project consists of multiple small tasks. The manager relies on an agent (she), who could be either an individual coder or a representative of a team, to complete these tasks. The agent is more productive at work when she is in a good mood. Only the agent knows her status. To motivate the agent, the manager needs to carefully design the reward and penalty scheme, and decide when to assign a task to the agent.[1] Such optimal project assignment problems are prevalent. For example, a firm hires an expert/headhunter to fill several positions; a team leader considers assigning a few tasks to a member; and a retailer needs to outsource some business to a service provider. The agent often has some private information: the headhunter is more informed of the difficulty level of finding suitable candidates; the team member knows whether she is feeling good or not; and the service provider has private information about the production cost. The principal's problem is that he is uncertain what the best course of action is.

This paper considers a dynamic mechanism design problem under the capacity constraint. A principal has multiple ($m \geq 1$) projects and needs to hire an agent to help with the projects. The agent can finish one project at each stage. The agent's cost is either high (c_H) or low (c_L), which is persistent and evolves stochastically over time. Hiring the agent to complete a project yields the principal a payoff. The principal could ask the agent to finish a project at every stage by paying a certain amount before all the projects are completed.[2] However, taking the agent's information into account and designing the mechanism strategically may result in payoff gains.

We aim to address two questions in this paper. First, we solve for the optimal dynamic mechanism in the setting with a general capacity constraint. It is shown that the optimal mechanism can be characterized by a sequence of deadlines. Second, we are particularly interested in settings with a large but finite number of projects. Intuitively, the setting with countably many projects provides a convenient idealization for the setting with a large finite number of projects. One may hope that the optimal mechanism in the setting with the capacity constraint would converge to that in the idealized model when the number of projects

[1] It is becoming popular nowadays, especially during the pandemic, to provide some form of flexibility for an agent when assigning the tasks. In particular, it is common to allow the agent to have flexible working arrangements depending on their willingness, rather than require her to finish a task every day. As reported in *The New York Times* (2018): "Corporate America is already catching up. Some 80 percent of companies now offer some form of flexible work arrangements, according to a 2015 survey by WorldatWork, a nonprofit human resources association, and FlexJobs, a career site."

[2] For example, a simple strategy is to pay the high cost (*i.e.*, c_H) to cover the agent's cost of taking a project.

increases. If this is true, then when the number of projects is sufficiently large, one could work with the case without the capacity constraint to avoid technical complications. We show that this asymptotic result does not hold. When the number of projects is finite, the principal's payoff is in general strictly bounded away from that in the setting without the capacity constraint.

To characterize the optimal mechanism, we identify a key intertemporal trade-off. Intuitively, when the principal considers whether to assign projects, he compares the gain from assigning them right away, with the payoff from holding and reassigning them in the future. In the optimal mechanism, (1) when delaying the assignment maximizes the total surplus (*i.e.*, efficient) given a high cost, the principal adopts the efficient assignment rule and is able to extract the full surplus; (2) when the delay is inefficient at a high cost, the principal needs to strategically delay the assignment by setting a sequence of deadlines to induce the agent to follow the truthtelling strategy. Thus, the optimal mechanism may exhibit inefficiency.

In the general setting with multiple projects, it is often complicated to identify the optimal mechanism. To illustrate the idea, we first discuss the simple case with one project. The principal's problem in this case is essentially an optimal stopping problem. If the cost c_H is sufficiently high, then it is efficient to hold the project and assign it at the subsequent stages whenever the agent is of a low cost. A simple mechanism for the principal is to assign the project and pay c_L to the agent if she reports a low cost, and to hold the project otherwise. This mechanism is efficient, incentive compatible, and the principal extracts all the surplus. The more subtle case is when it is efficient to assign projects even at a high cost. Certainly, the principal could make his decisions based on the prior belief without soliciting reports, but this may not be optimal. If the principal simply pays the agent the reported cost, then it is not incentive compatible for the agent to follow the truthtelling strategy. In the optimal mechanism, the principal uses delay as an instrument to screen the agent, and the duration of delay is carefully chosen so that the principal maximizes his payoff. As a result, the contract will exhibit inefficiency, as the project assignment may be distorted at later contracting stages.

The results in the one-project case can be extended to the general setting with multiple projects. Specifically, we show in Proposition 3 that the principal shall fix a non-increasing sequence of deadlines $\{T_k^*\}$. In the optimal mechanism, a low-cost agent is always allocated with projects. A high-cost agent is never allocated with a project if it is efficient to do so. Otherwise, for each k, the principal holds the k-th to last project until the deadline T_k^* is reached if the agent keeps reporting c_H.

An important property of the optimal mechanism is that the deadline has a stationary structure; that is, T_k^* does not depend on m – the number of initial projects. To understand this feature, note that postponing the assignment of a project delays not only the assignment of the current project, but also the assignment of all the subsequent projects. When there are k projects, intuitively, if the principal holds the current project and mimics the assignment rule for the next $k-1$ projects, then the earliest time at which he can reassign the holding project is k stages later. As a result, the trade-off he would consider is to compare the gain from assigning the project right away, from withholding and reassigning the project k-stages later. However, this consideration only takes into account the number of available projects, and has nothing to do with the previous assignment and reporting histories. That is, when determining deadlines, the principal is looking forward rather than looking back.

When the number of projects is large, identifying the optimal mechanism and calculating the principal's payoff can be hard. We consider the limit of the principal's payoff by taking the number of initial projects to infinity. It is natural to conjecture that the principal's limit payoff would be the same as that in the idealized setting without the capacity constraint.[3] However, it is shown in Proposition 5 that there is some dissonance between the idealized model and its asymptotic version. Specifically, the limit of the principal's payoff is in general strictly below that in the idealized setting. Besides, the principal may hold the projects forever in the setting with the capacity constraint, while every project gets eventually assigned in the model without the capacity constraint. Loosely, when the capacity constraint is present, the assignment is further delayed and the principal has a lower payoff.

The rest of the paper is organized as follows. The remainder of this section provides a review of the related literature. Section 2 introduces the model. Section 3 considers the one-project case in the complete-information and incomplete-information settings, and identifies the key conditions of the paper. Section 4 characterizes the optimal mechanism in the general environment, and provides further discussions. Section 5 compares the principal's payoffs in the models with and without the capacity constraint. Section 6 concludes. The Appendix collects some proofs.

Related Literature.

This paper joins the line of research in dynamic mechanism design, which considers a principal-agent problem with the agent's preference evolving stochastically over time; see, for example, Baron and Besanko (1984), Besanko (1985), Courty and Li (2000), Battaglini (2005), Esõ and Szentes (2007) and Pavan, Segal and

[3]The long-term contracting problem without the capacity constraint in our setting is similar to that in Battaglini (2005).

Toikka (2014). The literature on dynamic mechanism design is too vast to be discussed in the context of this paper, we refer the readers to the recent surveys by Pavan (2017) and Bergemann and Välimäki (2019) for more details.[4]

The most related work is the important paper by Battaglini (2005). He considers a nonlinear pricing model in which the buyer's valuation evolves over time according to a commonly known first-order Markov process with two states.[5] As explained above, the no-capacity constraint version of our model can be viewed as a similar problem studied in this paper – a dynamic contracting setting with infinitely many projects. Much of the work following Battaglini's paper focused on the case where the allocation problem is assumed to be time-invariant in the sense that the set of feasible choices at stage t does not depend on the allocation decisions at previous stages. This assumption is often violated in dynamic environments with capacity constraints. The key difference in our set-up is that we introduce an additional constraint on the principal side. The optimal mechanism therefore may or may not end in finite stages in our setting, while the interaction between the principal and the agent continues for infinite stages in Battaglini (2005). An important difficulty, new in this paper, is to characterize the key conditions that determine whether or not the principal should assign projects along the worst path, and to identify the deadline for each project. The working paper version of Garrett (2016) derived the optimal mechanism for a principal selling a durable good to an agent with varying values in the continuous-time setting. Result 4 therein is related to our one-project case, which shows that though the principal shall strategically delay, the project gets eventually assigned. In the current paper, we identify a sharp condition to determine whether the project will be assigned or not, and extend the one-project case to the multi-unit setting.

Another related strand considers the dynamic allocation problem with a fixed capacity of indivisible goods to sell by a (possibly infinite) deadline. This line of research typically considers the setting where potential buyers arrive randomly and the population of privately informed buyers changes over time. Gershkov and Moldovanu (2009) studied a revenue-maximizing monopolist selling several

[4]The theory of optimal dynamic mechanism design has been significantly developed in the last decade in various environments: *e.g.*, Garrett (2016) showed that the optimal price path fluctuates in durable good markets; Armstrong and Zhou (2015) considered a search market and showed how a seller may deter buyers from searching for a better product; and Li and Shi (2017) studied how a seller can disclose additional information to the buyer about her valuation without observing the realization. From a methodological standpoint, the analysis largely relies on the first-order approach. Pavan, Segal and Toikka (2014) provided a general treatment of this approach in the dynamic environment, and obtained necessary results for incentive compatibility. Battaglini and Lamba (2019) extended the analysis of the general model to settings where the first-order approach may not hold.

[5] Battaglini (2007) shows that the optimal contract with commitment is renegotiation proof under general conditions.

heterogeneous objects to short-lived agents who arrive sequentially, where the arrival of an agent is public information. When the arrivals are private information, Board and Skrzypacz (2016) considered the sales problem of finitely many identical units, and showed that the optimal selling mechanism is a deterministic sequence of posted prices. Gershkov, Moldovanu and Strack (2017) extended the model to cover the case where the buyer's arrival process is initially unknown. Each buyer is assumed to have a unit demand and the buyer's privately known valuation does not change over time in those papers. In the current paper, we assume that the agent can take multiple projects and has stochastically evolving costs.[6]

2 Model

Our model adds the capacity constraint to an otherwise standard dynamic screening setting. A principal has $m \geq 1$ projects, and needs to hire an agent to help with those projects. At each stage, the agent is able to finish one project at some cost. The cost at stage t is taken from the set: $\{c_H, c_L\}$ ($c_H > c_L > 0$). At the initial stage, the principal's prior belief is

$$P(c_1 = c_H) = \lambda_H \in (0,1) \quad \text{and} \quad P(c_1 = c_L) = \lambda_L = 1 - \lambda_H.$$

We assume that the distribution of the agent's cost at stage $t+1$ depends only on her cost at stage t, meaning that the agent's cost c_t at stage-t is a sufficient statistic for her later costs. Denote

$$P[c_{t+1} = c_H | c_t = c_H] = \alpha_H \quad \text{and} \quad P[c_{t+1} = c_H | c_t = c_L] = \alpha_L.$$

Costs are assumed to be persistent in the sense that $0 < \alpha_L \leq \alpha_H < 1$. That is, the cost process satisfies the first-order stochastic dominance: the probability of a high cost tomorrow conditional on today's cost being high is higher than the probability of a high cost tomorrow conditional on today's cost being low. We adopt the following notations: a sequence of costs from stage 1 to stage t is denoted by $c^t = (c_1, \ldots, c_t)$, and $\{c^0\} = \emptyset$. Throughout this paper, we assume that the cost is the agent's private information.

At any stage, the principal could assign a project to the agent. The payoff of the principal for a completed project is v ($v > c_H > c_L$). To incentivize the agent to truthfully report her private costs, the principal can choose a monetary transfer to the agent. Both the principal and the agent are impatient and have a discount

[6] Garrett (2017) considered a sales problem of non-durable goods where the buyer arrives privately and randomly with evolving private valuations. This paper does not consider the issue of capacity constraints.

factor $\delta \in (0,1)$.

We use s_t to track the number of projects left unfinished at stage t. That is, if there are m projects at stage 1, then $s_1 = m$. If there are m' ($m' \leq m$) projects left at stage t, then $s_t = m'$. If the principal does not assign a project to the agent at stage t, then the number of unfinished projects remains the same at the next stage: $s_{t+1} = s_t$. Otherwise, $s_{t+1} = s_t - 1$, implying that one project has been taken by the agent. For $t \geq 1$, $s^t = (s_1, \ldots, s_t)$.

Mechanism.

At the first stage, the principal can fully commit to a long-term contract. By the revelation principle, it is without loss to focus on the incentive compatible direct mechanisms. A direct mechanism $\Gamma = (\mathbf{q}, \mathbf{p})$ is a collection of assignment rules $\mathbf{q} = \{q_t\}_{t \geq 1}$ and payment rules $\mathbf{p} = \{p_t\}_{t \geq 1}$. The timing is as follows:

1. At stage 1, a private cost $c_1 \in \{c_H, c_L\}$ is randomly drawn. The agent is then asked to make a report $\tilde{c}_1 \in \{c_H, c_L\}$. Based on the report, the principal assigns a project to the agent with probability $q_1(\tilde{c}_1; s_1) \in [0,1]$ and chooses a monetary transfer $p_1(\tilde{c}_1; s_1) \in \mathbb{R}$, where $s_1 = m$.
2. At stage $t \geq 1$, if a project is completed, then $s_{t+1} = s_t - 1$; otherwise, $s_{t+1} = s_t$.
3. If $s_t \geq 1$ at stage $t > 1$, then the agent draws a private cost $c_t \in \{c_H, c_L\}$ following the law of motion $P[c_t | c_{t-1}]$, and is further asked to make a report $\tilde{c}_t \in \{c_H, c_L\}$. Based on the history $\tilde{c}^t = (\tilde{c}^{t-1}, \tilde{c}_t)$ and $s^t = (s^{t-1}, s_t)$, the probability of assigning a project at that stage is $q_t(\tilde{c}^t; s^t) \in [0,1]$, and a transfer $p_t(\tilde{c}^t; s^t) \in \mathbb{R}$ is made.[7]
4. If $s_t = 0$ for $t > 1$, then $q_t(\tilde{c}^t; s^t) \equiv 0$ and $p_t(\tilde{c}^t; s^t) \equiv 0$ for any $\tilde{c}^t$. That is, all the projects have been completed and the contract is terminated automatically.

The agent's problem.

At stage t, given the assignment probability q_t and the transfer p_t, the stage payoff of the agent is

$$-q_t \cdot c_t + p_t.$$

Fix a mechanism $\Gamma = (\mathbf{q}, \mathbf{p})$. At stage $t \geq 1$, given the history c^{t-1} and s^t, and the

[7]In the direct mechanism, the assignment rule and the payment rule depend on both the reporting history and the history of the tracking numbers of projects. This is because the principal cares about not only the agent's costs, but also the number of the unfinished projects.

cost c_t at the current stage, the agent's expected payoff is

$$U(c^{t-1}, c_t; s^t) = \sum_{i \geq 0} \delta^i \mathbb{E}\big[u(c^{t+i}; s^{t+i})|c_t\big],$$

where

$$u(c^{t+i}; s^{t+i}) = -q_{t+i}(c^{t+i}; s^{t+i})c_{t+i} + p_{t+i}(c^{t+i}; s^{t+i}).$$

Given s^t, let $\tilde{s}^{t+1} = (s^t, s_t - 1)$ be a vector of the tracking numbers, where $\tilde{s}_{t+1} = s_t - 1$, meaning that one project has been assigned in the end of stage t. On the other hand, let $\hat{s}^{t+1} = (s^t, s_t)$, meaning that there is no assignment at stage t and $\hat{s}_{t+1} = s_t$.

For any $t \geq 1$, the agent is called *incentive compatible* at stage t (IC_t) if for any c^{t-1}, c_t, $\tilde{c}_t$, and s^t,

$$\begin{aligned}
&u(c^t; s^t) + \delta q_t(c^t; s^t)\mathbb{E}\left[U(c^t, c_{t+1}; \tilde{s}^{t+1})|c_t\right] \\
&+ \delta(1 - q_t(c^t; s^t))\mathbb{E}\left[U(c^t, c_{t+1}; \hat{s}^{t+1})|c_t\right] \\
&\geq u(c^{t-1}, \tilde{c}_t; s^t) + \delta q_t(c^{t-1}, \tilde{c}_t; s^t)\mathbb{E}\left[U(c^{t-1}, \tilde{c}_t, c_{t+1}; \tilde{s}^{t+1})|c_t\right] \\
&+ \delta(1 - q_t(c^{t-1}, \tilde{c}_t; s^t))\mathbb{E}\left[U(c^{t-1}, \tilde{c}_t, c_{t+1}; \hat{s}^{t+1})|c_t\right].
\end{aligned}$$

The left hand side of the inequality is the agent's payoff if she truthfully reports her cost c_t at stage t. It consists of three parts. The first part $u(c^t; s^t)$ is the stage payoff of the agent. The second part $\delta q_t(c^t; s^t)\mathbb{E}\left[U(c^t, c_{t+1}; \tilde{s}^{t+1})|c_t\right]$ is the discounted future payoff if one project is assigned (with probability $q_t(c^t; s^t)$), while the third term $\delta(1 - q_t(c^t; s^t))\mathbb{E}\left[U(c^t, c_{t+1}; \hat{s}^{t+1})|c_t\right]$ is the discounted future payoff if no project is assigned (with probability $1 - q_t(c^t; s^t)$). The agent's overall payoff is the summation of the three parts. The right hand side is her payoff if she misreports $\tilde{c}_t$. The agent is said to be incentive compatible if it is optimal for her to choose truthtelling at any stage.

For $t \geq 1$, the agent is said to be *individually rational* at stage t (IR_t) if for any c^t and s^t,

$$U(c^t; s^t) \geq 0.$$[8]

That is, the agent has the choice to walk away from the contract at any time if the expected continuation payoff falls below the reservation value 0.

The principal's problem.

Given that the agent truthfully reports her private costs, the expected payoff

[8]The $\{IC_t\}_{t\geq 1}$ and $\{IR_t\}_{t\geq 1}$ constraints are satisfied automatically if $s_t = 0$.

of the principal is

$$\mathbb{E}\left[\sum_{t\geq 1}\delta^{t-1}(vq_t(c^t;s^t)-p_t(c^t;s^t))\right].$$

The principal's aim is to maximize his expected payoff subject to all (IC_t) and (IR_t) constraints for $t \geq 1$.

3 Illustration: one project

We first work with the setting in which the principal has only one project ($m = 1$), and illustrate several key points of the main result via this simple case.

Benchmark: Complete information

We start with the benchmark case where the principal has complete information; that is, the principal observes the agent's costs. This assumption simplifies the analysis by omitting the agent's incentive problem.[9] The question is then reduced to an individual decision problem: When shall the principal assign the project to the agent? Since the principal knows the cost, he simply pays the agent her cost whenever the agent is asked to work on the project. As a result, the principal's aim is to maximize the surplus.

If the agent has a low cost, it is in the principal's interest to assign the project to the agent. The trade-off would arise if the agent has a high cost. In this case, the principal needs to compare the surplus between two possible options:

1. If the project is assigned immediately, then the surplus is $v - c_H$.
2. If the principal chooses not to assign the project at the current stage, then due to the stationary structure, it would not be optimal to assign the project at c_H at any later stage. That is, the principal shall wait until the cost is low. The surplus is

$$\sum_{t\geq 0}\delta^{t+1}P(c_H|c_H)^t P(c_L|c_H)(v-c_L) = \frac{\delta(1-\alpha_H)}{1-\delta\alpha_H}(v-c_L).$$

We have

$$v - c_H \leq \frac{\delta(1-\alpha_H)}{1-\delta\alpha_H}(v-c_L)$$

if and only if

$$v - c_H \leq \delta\Big((v-c_L)(1-\alpha_H)+(v-c_H)\alpha_H\Big).$$

[9] The principal still needs to incentivize the agent to participate in the mechanism.

The following condition plays a key role in deriving the results of this paper.

Condition 1.

a. $v - c_H \leq \delta\Big((v - c_L)(1 - \alpha_H) + (v - c_H)\alpha_H\Big)$.

b. $v - c_H > \delta\Big((v - c_L)(1 - \alpha_H) + (v - c_H)\alpha_H\Big)$.

The next proposition is straightforward.

Proposition 1. *The principal's optimal mechanism under complete information is characterized by the following assignment rule.*

1. *For any c^{t-1} and s^t such that $s_t = 1$, let*

$$q_t(c^{t-1}, c_L; s^t) = 1.$$

2. *Under Condition 1 (a), for any s^t such that $s_t = 1$,*

$$q_t(c_H^t; s^t) = 0,$$

where c_H^t is a vector with t components of c_H. That is, the principal never assigns the project along the path $(c_H, c_H, \ldots)$.

3. *Under Condition 1 (b),*

$$q_1(c_H; s^1) = 1.$$

That is, the principal assigns the project at the initial stage.

We say an assignment rule is efficient if it maximizes the expected total surplus. The above proposition shows that for an efficient assignment rule, the project is assigned when a low cost arrives. Given the path with only high costs, the assignment could happen either immediately or never, depending on whether or not the future expected surplus following a high cost is higher than $v - c_H$, the surplus at the current stage if the project is assigned immediately.

Incomplete information

Hereafter, we study the optimal mechanism for the case where the principal cannot observe the agent's costs. When there is only one project, the decision for the principal is easy if the agent is competent (with low cost c_L): the principal shall assign the project to the agent immediately (as in the complete information setting). However, when the agent has a high cost c_H, the principal faces an intertemporal trade-off: he can either assign the project to the agent at that stage, or hold the project and wait for the agent's cost to become low. While this comparison is familiar from the complete information setting, the analysis is more complicated and the optimal mechanism is considerably different.

We show that the principal sets a deadline T_1^{1*} in the optimal mechanism. At any stage before the deadline T_1^{1*}, the principal assigns the project to the agent only if the agent reports a low cost at that stage. If the agent keeps reporting "c_H", then the principal shall wait until the deadline, and assign the project at stage T_1^{1*} regardless of the agent's report. This deadline T_1^{1*} might be finite or infinite, depending on the comparison between the current surplus and future expected surplus (*i.e.*, Condition 1 (a) or (b)).

The following result is useful by showing how the agent's expected continuation payoff depends on her cost at the current stage.[10]

Lemma 1. *Fix an incentive compatible and individually rational mechanism* Γ *with the assignment rule* $\mathbf{q}$*. Given the history* c^{t-1} *and* s^t *at stage* $t \geq 1$*, we have*

$$\begin{aligned} & U(c^{t-1}, c_L; s^t) - U(c^{t-1}, c_H; s^t) \\ \geq\ & (c_H - c_L) \sum_{i=0}^{\infty} \delta^i (\alpha_H - \alpha_L)^i q_{t+i}(c^{t-1}, c_H^{i+1}; s^{t+i}) \prod_{1 \leq j \leq i} [1 - q_{t+j-1}(c^{t-1}, c_H^j; s^{t+j-1})]. \end{aligned} \tag{1}$$

Remark 1. *The inequality above gives a lower bound for the additional payoff that the agent could enjoy when the cost is low. At stage* t*, in order for the low-cost agent not to misreport a high cost, the agent's payoff from truthtelling must be no less than the payoff from misreporting. This results in the following inequality:*

$$\begin{aligned} & U(c^{t-1}, c_L; s^t) - U(c^{t-1}, c_H; s^t) \\ \geq\ & (c_H - c_L) q_t(c^{t-1}, c_H; s^t) \\ & + \delta(1 - q_t(c^{t-1}, c_H; s^t))(\alpha_H - \alpha_L)[U(c^{t-1}, c_H, c_L; \hat{s}^{t+1}) - U(c^{t-1}, c_H, c_H; \hat{s}^{t+1})], \end{aligned}$$

where $\hat{s}^{t+1} = (s^t, s_t)$*. The right hand side is a convex combination of two terms. The first term is the additional surplus* $c_H - c_L$ *for having low cost rather than high cost, if the project is assigned at the current stage. The second term is the difference of the expected continuation payoffs based on low cost and high cost at stage* t*, multiplied by the probability that the project is not assigned. Applying the same inequality to the difference* $U(c^{t-1}, c_H, c_L; \hat{s}^{t+1}) - U(c^{t-1}, c_H, c_H; \hat{s}^{t+1})$ *repeatedly, we get the right hand side of Inequality (1).*

As is standard in the literature, we show that in the optimal mechanism, the expected continuation payoff of the agent with cost c_H *is* 0*. In addition, Inequality (1) is binding. The term* $\delta^i(\alpha_H - \alpha_L)^i(c_H - c_L)$ *can be understood as the expected rent of having low cost rather than high cost at stage* t*, if the*

[10] The proofs of Lemma 1 and Proposition 2 are left in Section 7.1.

project is assigned at stage $t+i$. The summation in Inequality (1) adds up those potential additional payoffs, with the parameter $q_{t+i}(c^{t-1}, c_H^{i+1}; s^{t+i}) \prod_{1 \leq j \leq i}[1 - q_{t+j-1}(c^{t-1}, c_H^{j}; s^{t+j-1})]$ being the probability that this assignment happens exactly i stages later. Thus, the lower bound is simply the aggregate of the expected payoffs from every subsequent stage.

Note that $U(c_H; s^1) \geq 0$ by the participation constraint. Given Inequality (1), the expected payoff of the agent is

$$\begin{aligned}
&\lambda_L U(c_L; s^1) + \lambda_H U(c_H; s^1) = \lambda_L (U(c_L; s^1) - U(c_H; s^1)) + U(c_H; s^1) \\
&\geq \lambda_L (c_H - c_L) \sum_{i=0}^{\infty} \delta^i (\alpha_H - \alpha_L)^i q_{i+1}(c_H^{i+1}; s^{i+1}) \prod_{1 \leq j \leq i} [1 - q_j(c_H^j; s^j)].
\end{aligned}$$

The expected surplus is

$$\begin{aligned}
&\sum_{c_1 \in \{c_H, c_L\}} \lambda(c_1) \Big\{ q_1(c_1; s_1)(v - c_1) \\
&+ \delta[1 - q_1(c_1; s_1)] \big(W(c_1, c_H; s_1, s_1) P(c_H | c_1) + W(c_1, c_L; s_1, s_1) P(c_L | c_1) \big) \Big\},
\end{aligned}$$

where $\lambda(c_1)$ is probability for the cost to be c_1 at stage 1, and $W(c^2; s^2)$ is the expected surplus at stage 2 given $(c^2; s^2)$. The payoff of the principal must be greater than the expected surplus less the lower bound of the agent's expected rents.

Lemma 2. *Given an incentive compatible and individually rational mechanism Γ with the assignment rule $\mathbf{q}$, the expected payoff of the principal is no greater than*

$$\begin{aligned}
E_P^1 = &\sum_{c_1 \in \{c_H, c_L\}} \lambda(c_1) \Big\{ q_1(c_1; s_1)(v - c_1) \\
&+ \delta[1 - q_1(c_1; s_1)] \big(W(c_1, c_H; s_1, s_1) P(c_H | c_1) + W(c_1, c_L; s_1, s_1) P(c_L | c_1) \big) \Big\} \\
&- \lambda_L (c_H - c_L) \sum_{i=0}^{\infty} \delta^i (\alpha_H - \alpha_L)^i q_{i+1}(c_H^{i+1}; s^{i+1}) \prod_{1 \leq j \leq i} [1 - q_j(c_H^j; s^j)].
\end{aligned}$$

Note that the expected surplus at the second stage is argumented by the probability $1 - q_1(c_1; s_1)$. That is, if $q_1(c_1; s_1) = 1$, then the mechanism will end at the first stage and the subsequent costs are irreverent to the surplus. If $q_1(c_1; s_1) < 1$, then the expected continuation surplus needs to be adjusted accordingly.

Next, we characterize the optimal mechanism for the one-project case in the proposition below.

Proposition 2. *The optimal mechanism is characterized by the following assignment rule. In addition, the principal's payoff is given by E^1_P in Lemma 2.*

1. *For any c^{t-1} and s^t such that $s_t = 1$, let*

$$q_t(c^{t-1}, c_L; s^t) = 1.$$

2. *Under Condition 1 (a), for any s^t such that $s_t = 1$,*

$$q_t(c^t_H; s^t) = 0.$$

That is, the principal never assigns the project along the path $(c_H, c_H, \ldots)$. Let $T^{1*}_1 = \infty$.

3. *Suppose that Condition 1 (b) holds. Denote T^{1*}_1 as the smallest t such that*

$$\begin{aligned} v - c_H \quad & \geq \delta\Big((v - c_L)(1-\alpha_H) + (v - c_H)\alpha_H\Big) \\ & + \frac{\lambda_L}{\lambda_H}(c_H - c_L)\Big[1 - \delta(\alpha_H - \alpha_L)\Big]\left(\frac{\alpha_H - \alpha_L}{\alpha_H}\right)^{t-1}. \end{aligned} \tag{2}$$

Let

$$q_t(c^t_H; s^t) = \begin{cases} 0, & t < T^{1*}_1, \\ 1, & t = T^{1*}_1. \end{cases}$$

Remark 2. *The above proposition characterizes the optimal mechanism via the assignment rule. In the optimal mechanism, Inequality (1) is binding and the principal's maximal payoff is E^1_P in Lemma 2.*

The optimal assignment rules in Proposition 2 (1-2) are the same as that in Proposition 1. Intuitively, the agent with a high cost would not misreport the low cost. Thus, the agent must be truthtelling if she reports c_L, implying that the assignment rule should be surplus maximizing given the report c_L. If Condition 1 (a) holds, then it is efficient not to assign the project given a high cost. The agent is incentive compatible and is left zero rent at both costs. It is optimal for the principal to adopt this assignment rule since he gets the total surplus. That is, the principal's interest is aligned with the goal of surplus maximization.

Suppose that Condition 1 (b) holds. Then efficiency, incentive compatibility, and full surplus extraction cannot be achieved at the same time. Otherwise, the agent may find it beneficial to report c_H when having c_L. The principal uses a delayed assignment rule along the path $(c_H, c_H, \ldots)$ so that the low-cost agent does not benefit from misreporting.

The proof proceeds by first identifying an assignment rule which maximizes the upper bound E_P^1. We then construct the corresponding payments such that the mechanism with the given assignment and payment rules satisfy all the incentive constraints and participation constraints. As a result, this mechanism gives the maximal payoff that the principal could achieve.

The upper bound E_P^1 is the expected surplus less the lower bound of the agent's expected rents. Note that only the assignment along the path $(c_H, c_H, \ldots)$ is relevant for calculating this lower bound. Whenever a low cost c_L is reported, the principal only needs to maximize the surplus. Thus, Claim (1) in Proposition 2 is straightforward: as the surplus $v - c_L$ (if the project is assigned immediately) is always higher than the discounted future surplus, the principal shall assign the project immediately.

To obtain Proposition 2 (2-3), we need to compare $v - c_H$ and $\delta\big((v - c_L)(1 - \alpha_H) + (v - c_H)\alpha_H\big)$. If the former is less than the latter, then it means that the discounted future surplus based on the low cost is more valuable than the surplus at this stage given the high cost. The principal finds it more attractive to delay at a high cost, and chooses never to assign the project along the path $(c_H, c_H, \ldots)$; this gives Proposition 2 (2). If $v - c_H$ is higher than $\delta\big((v - c_L)(1 - \alpha_H) + (v - c_H)\alpha_H\big)$, then Proposition 2 (3) characterizes the (finite) optimal stopping stage T_1^{1*}.[11]

One way to understand the assignment rule in Proposition 2 (2-3) is as follows. Along the path $(c_H, c_H, \ldots)$, the principal needs to decide at which stage the project should be assigned. This is an optimal stopping problem for the principal. Suppose that the principal chooses some $t' \geq 1$, and lets $q_t(c_H^t; s^t) = 0$ for any $t < t'$ and $q_{t'}(c_H^{t'}; s^{t'}) = 1$. Then the relevant part of E_P^1 for the principal is

$$\begin{aligned}
&\lambda_H \delta \sum_{k\geq 0}^{t'-2} [\delta^k P(c_H|c_H)^k] P(c_L|c_H)(v - c_L) + \lambda_H \delta^{t'-1} P(c_H|c_H)^{t'-1}(v - c_H) \\
&\quad - \lambda_L (c_H - c_L) \delta^{t'-1} \left(P(c_H|c_H) - P(c_H|c_L)\right)^{t'-1} \\
&= \lambda_H \delta (1 - \alpha_H)(v - c_L) \frac{1 - \delta^{t'-1}\alpha_H^{t'-1}}{1 - \delta\alpha_H} + \lambda_H \delta^{t'-1} \alpha_H^{t'-1}(v - c_H) \\
&\quad - \lambda_L (c_H - c_L) \delta^{t'-1} \left(\alpha_H - \alpha_L\right)^{t'-1}. \qquad (3)
\end{aligned}$$

The first term describes the expected surplus in the case where the initial cost is c_H and it changes to c_L at no later than stage t'. The second term is the part of the expected surplus for the case where the cost at the first t' stages are all "c_H". The last term is the rent left to the agent. If one chooses $t' = \infty$, then the payoff

[11] In particular, as $t \to \infty$, the right-hand side of Inequality (2) approaches $\delta\big((v - c_L)(1 - \alpha_H) + (v - c_H)\alpha_H\big)$, implying that Inequality (2) must be satisfied for sufficiently large t.

in (3) is reduced to be

$$\lambda_H \delta(1-\alpha_H)(v-c_L)\frac{1}{1-\delta\alpha_H}. \tag{4}$$

It is easy to check that (3) $\leq$ (4) for any t' if and only if Condition 1 (a) holds, resulting in Proposition 2 (2). If Condition 1 (b) holds, we show that the payoff in (3) is single peaked in the stopping stage t'. Thus, a (finite) optimal stopping stage exists, which is characterized in Proposition 2 (3).

In summary, identifying the assignment choice along the path $(c_H, c_H, \ldots)$ in Proposition 2 (2-3) involves two considerations. The principal needs to guarantee that given an assignment at some stage, (a) the difference between the surplus and the agent's rent is nonnegative, (b) the chosen stage to assign the project is optimal.

4 The general setting

In this section, we analyze the principal's problem subject to the general capacity constraint: $m \geq 1$. The optimal mechanism is characterized via the assignment rule in Proposition 3. It is shown that when the cost is low, the principal shall follow the efficient assignment rule and assign a project to the agent at that stage. When the agent's cost is high, however, inefficiency may arise in the optimal mechanism. The key trade-off for the principal is to compare the payoff from assigning the projects right away with the payoff from reassigning them in the future. The principal uses delay as an instrument to screen the agent. Compared with the one-project case, the problem here is more complicated. In particular, postponing the assignment of one project not only induces a delay of this project, but also delays the assignments of all the subsequent projects. With this complication, we show that the optimal length of delays can be pinned down by a sequence of deadlines $\{T_k^{m*}\}_{1\leq k\leq m}$. which are defined following the reverse order. That is, T_m^{m*} is the deadline for the first project to be assigned, T_{m-1}^{m*} is the deadline for the second project to be assigned, etc.[12] Two features of the optimal mechanism are highlighted in Proposition 4.

1. The sequence $\{T_k^{m*}\}_{1\leq k\leq m}$ is decreasing in k. This is intuitive given the reverse order.

2. The deadlines satisfy certain stationary structure. Specifically, the deadlines

[12]The proof of Proposition 3 is left in Appendix.

are immune to the number of original projects: for any $m \geq k$,

$$T_k^{m*} = T_k^{(m+1)*}.$$

Further discussions of the results are provided in Section 4.2.

4.1 The optimal mechanism

We first extend the definition of the discounted future surplus to the general setting. Let

$$f_k(c_1) = \delta^{k-1} \sum_{c_2,\ldots,c_k \in \{c_H, c_L\}} \left[(v - c_k) \cdot \prod_{1 \leq j \leq k-1} P(c_{j+1}|c_j) \right].$$

Note that

$$f_1(c_1) = v - c_1.$$

The term $f_k(c_1)$ is the discounted expected surplus $k-1$ stages later when the initial cost is c_1 for $k \geq 1$.

The following proposition characterizes the optimal mechanism in the general setting.

Proposition 3. *The optimal mechanism is characterized by the following assignment rule.*

1. *For any c^{t-1} and s^t such that $s_t \geq 1$, let*

$$q_t^*(c^{t-1}, c_L; s^t) = 1.$$

2. *Under Condition 1 (a), for any c^{t-1} and s^t such that $s_t \geq 1$,*

$$q_t^*(c^{t-1}, c_H; s^t) = 0.$$

3. *Suppose that Condition 1 (b) holds.*

   (a) *For any $c^{t-1} \neq c_H^{t-1}$ and s^t such that $s_t \geq 1$,*

$$q_t^*(c^{t-1}, c_H; s^t) = 1.$$

   (b) *For $1 \leq k \leq m$, denote T_k^{m*} as the smallest t such that*

$$\begin{aligned} v - c_H \quad & \geq \delta\Big(f_k(c_L)(1-\alpha_H) + (v - c_H)\alpha_H\Big) \\ & + \frac{\lambda_L}{\lambda_H}(c_H - c_L)\Big[1 - \delta(\alpha_H - \alpha_L)\Big]\left(\frac{\alpha_H - \alpha_L}{\alpha_H}\right)^{t-1}. \end{aligned} \tag{5}$$

Then for $c^t = c_H^t$ and s^t such that $s_t = k \geq 1$, let

$$q_t^*(c_H^t; s^t) = \begin{cases} 0, & t < T_k^{m*}; \\ 1, & t \geq T_k^{m*}. \end{cases}$$

That is, the principal shall set a deadline T_k^{m*} for the k-th to last project, and wait until stage T_k^{m*} to assign that project along the path $(c_H, c_H, \ldots)$.

Two important properties of the deadlines in Proposition 3 (3b) are summarized in the proposition below.

Proposition 4. *Suppose that Condition 1 (b) holds.*

1. *For $1 \leq k \leq m$, the deadline T_k^{m*} does not depend on m.*
2. *For any $c \in \{c_H, c_L\}$, $\{f_k(c)\}_{k \geq 1}$ is a decreasing sequence and converges to 0 as $k \to \infty$, implying that the sequence $\{T_k^{m*}\}_{k \geq 1}$ is decreasing in k.*

Proof. (1) By Inequality (5), T_k^{m*} does not depend on m for any k.

(2) It is clear that

$$v - c_L > \delta\big((v - c_L)(1 - \alpha_L) + (v - c_H)\alpha_L\big).$$

By Condition 1 (b),

$$v - c_H > \delta\Bigg((v - c_L)(1 - \alpha_H) + (v - c_H)\alpha_H\Bigg)$$

Iterating these two inequalities, $\{f_k(c)\}$ is a decreasing sequence. As $k \to \infty$, δ^k converges to 0, which implies that $f_k(c)$ converges to 0 for any $c \in \{c_H, c_L\}$. Since $\{f_k(c_L)\}$ is decreasing in k, so is the sequence $\{T_k^{m*}\}$. □

4.2 Remarks

By Proposition 4, T_k^{m*} does not depend on m. Thus, we can simplify the notation by omitting the superscript m, and write T_k^{m*} as T_k^*.

To illustrate the deadlines $\{T_k^*\}_{k \geq 1}$ in Propositions 3 and 4, we present an example below.

Example 1. *Suppose that $v = 2$, $\delta = \frac{9}{10}$, $c_H = 1$, $c_L = \frac{1}{2}$, $\alpha_H = \frac{9}{10}$, $\alpha_L = \frac{1}{10}$, $\lambda_H = \frac{1}{10}$, and $\lambda_L = \frac{9}{10}$. It is easy to check that Condition 1 (b) holds. Inequality (5) in Proposition 3 is reduced to: for $k \geq 1$, T_k^* is the smallest t such that*

$$19 \geq 9 f_k(c_L) + 126 \left(\frac{8}{9}\right)^{t-1}.$$

Note that

$$f_1(c_L) = v - c_L = \frac{3}{2},$$

$$f_2(c_L) = \delta[(v - c_H)\alpha_L + (v - c_L)(1 - \alpha_L)] = \frac{261}{200},$$

$$\ldots,$$

and

$$f_k(c_L) \to 0 \text{ as } k \to \infty.$$

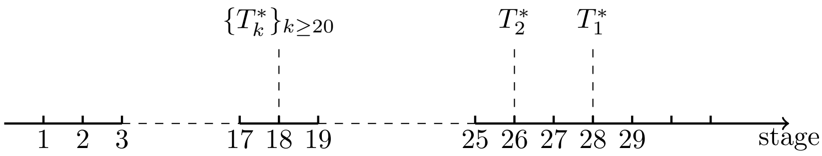

Figure 1: Deadlines

We have that

$$T_1^* = 28, \qquad T_2^* = 26, \ldots,$$

and

$$T_k^* = 18 \quad \text{for any } k \geq 20.$$

The sequence $\{T_k^*\}_{k\geq 1}$ *is shown in Figure 1.*

Remark 3 below discusses the efficiency issue for the optimal mechanism in Proposition 3.

Remark 3. *In Proposition 3 (1-2), the assignment rule is efficient. The intuition is similar to that behind Proposition 2 (1-2). When Condition 1 (b) holds, however, there could be two possible cases to consider.*

Along the path c^{t-1} *that is not* c_H^{t-1}*, the principal assigns the projects right away. One can imagine that after the agent reveals herself to have a low cost, there is no residual asymmetric information. Before the realization of the cost at the following stage, the principal can delegate all the projects to the agent so that the agent directly benefits from the completion of those projects. The principal simply pays the agent his expected payoff before this delegation. Along the path* $c^{t-1} = c_H^{t-1}$*, the principal shall delay the assignment of the projects to deter the* *agent from lying, subject to the deadlines determined in Inequality (5).*[13]

Note that if Condition 1 (b) holds, then it is efficient to assign the projects right away regardless of the cost. As a result, though the assignment rule in (3a) is efficient, inefficiency appears in (3b). However, since the sequence $\{T_k^{m*}\}$ is decreasing in k by Proposition 4, the assignment rule must be efficient after the furthest deadline T_1^{m*} in (3b).[14] *It implies that the optimal mechanism will be efficient in the long run.*[15]

The following remark explains the intuition for deriving Inequality (5).

Remark 4. *To understand Inequality (5), note that the only difference between Inequalities (2) and (5) is that in the latter inequality, $v - c_L$ (i.e., $f_1(c_L)$) is replaced by $f_k(c_L)$.*

Recall that in the one-project case, to derive Proposition 2 (3), the principal compares the surplus at the current stage given a high cost (i.e., $v - c_H$) with the future surplus based on a low cost.

In the multi-project case, suppose that there are k projects left. If the principal assigns the first project to the agent given a high cost, then his payoff consists of two parts. The first part comes from the surplus $v - c_H$ at the current stage. The second part is from the future surplus based on the next $k - 1$ projects. Alternatively, the principal can hold the current project and mimic the assignment rule for the next $k - 1$ projects. The earliest time at which he can reassign the project is k stages later. The principal's payoff also consists of two parts. The second part remains the same, while the first part is $f_k(c_L)$, the expected surplus from the holding project. Thus, the principal shall compare $v - c_H$ with $f_k(c_L)$, resulting in Inequality (5).

In the following remark, we shall discuss possible payment rules.

Remark 5. *It is clear that there could be many payment schedules that are compatible with the assignment rule in Proposition 3, as any two payment rules with the same present value can give the same incentives to the agent. As discussed in Remark 3, a simple payment rule is simply delegating all the projects to the low-cost agent and paying her the expected payoff. To make sure the agent will not*

[13]It is possible that the deadline T_k^{m*} has already been passed when it is the first time that the principal has k projects. For example, imagine that the principal has four projects to finish and the deadline T_2^{4*} for the second to last project is 2. Then the earliest possible stage for the principal to assign the second to last project is the third stage. Since the deadline T_2^{4*} must have been passed when the principal has the opportunity to assign that project, he just assigns it to the agent regardless of the report whenever he has a chance to do it.

[14]By definition, the principal assigns a project at each stage after stage T_1^{m*} regardless of the cost.

[15]When Condition 1 (b) holds, the optimal mechanism in Proposition 3 is characterized by the same two properties as in Battaglini (2005), namely "generalized no distortion at the top" and "vanishing distortions at the bottom". When Condition 1 (a) holds, the allocation is always efficient and there is no distortion in our setting.

walk away from the contract after receiving the payment, the principal can commit to making the anticipated payment after all the projects are finished.

5 The limit case

When there is a large number of projects, one may ask whether the principal's optimal payoff in the setting with the capacity constraint (*i.e.*, m is finite) is close to the optimal payoff in the setting without the capacity constraint (*i.e.*, $m = +\infty$). In this section, we show in Proposition 5 that such an asymptotic result may not hold. We present a "discontinuity at infinity" result by showing that the limit of the principal's optimal payoff in the former case could be strictly lower than the optimal payoff in the latter case. Thus, the capacity constraint may hamper the principal's ability to extract rent.

We summarize the results in the setting without the capacity constraint in the following lemma; see Battaglini (2005) for more details.

Lemma 3 (Battaglini (2005))**.** *Fix an incentive compatible and individually rational mechanism* Γ *with the assignment rule* $\mathbf{q}^\infty$*.*

1. *The optimal mechanism is characterized by the following assignment rule.*
   (a) *For any* c^{t-1}*, let*
   $$q_t^\infty(c^{t-1}, c_L) = 1.$$
   (b) i. *For any* $c^{t-1} \neq c_H^{t-1}$*,*
   $$q_t^\infty(c^{t-1}, c_H) = 1.$$
   ii. *Denote* T_∞ *as the smallest* t *such that*
   $$v - c_H \geq \frac{\lambda_L}{\lambda_H}(c_H - c_L)\left(\frac{\alpha_H - \alpha_L}{\alpha_H}\right)^{t-1}. \tag{6}$$
   Then for $c^t = c_H^t$*, let*
   $$q_t^\infty(c_H^t) = \begin{cases} 0, & t < T_\infty; \\ 1, & t \geq T_\infty. \end{cases}$$
   That is, T_∞ *is the deadline for the first assignment to happen along the path* $(c_H, c_H, \ldots)$*. If* c_L *has appeared before or the deadline* T_∞ *has been reached, then the principal will assign a project to the agent at each stage regardless of the cost.*

2. *Given the history c^{t-1} at stage $t \geq 1$, we have*

$$\begin{aligned} & U(c^{t-1}, c_L) - U(c^{t-1}, c_H) \\ &= (c_H - c_L) \sum_{i=0}^{\infty} \delta^i \left(\alpha_H - \alpha_L\right)^i q^{\infty}_{t+i}(c^{t-1}, c_H^{i+1}). \end{aligned}$$

Let E^{∞}_A be the summation on the right hand side of the inequality above.

3. *The expected payoff of the principal is*

$$\begin{aligned} E^{\infty}_P = & \sum_{c_1 \in \{c_H, c_L\}} \lambda(c_1) \Big\{ q^{\infty}_1(c_1)(v - c_1) \\ & + \delta \left(W(c_1, c_H) P(c_H | c_1) + W(c_1, c_L) P(c_L | c_1) \right) \Big\} - \lambda_L \cdot E^{\infty}_A. \end{aligned}$$

In the proposition below, we show that the assignment of the projects in the setting with the capacity constraint is in general further delayed compared with the assignment in the setting without the capacity constraint, making the principal's payoff in the former case strictly less than that in the latter case.

Proposition 5. *The sequence $\{E^{m*}_P\}$ is increasing in m. Let $E^{\infty*}_P$ and T^*_∞ be the limits of the sequences $\{E^{m*}_P\}$ and $\{T^*_m\}$, respectively. We have that*

$$E^{\infty*}_P \leq E^{\infty}_P, \qquad T^*_\infty \geq T_\infty.$$

Before giving the proof, we revisit Example 1 to illustrate the comparison in Proposition 5.

Example 2 (Revisiting Example 1)**.** *In Example 1, $T^*_k = 18$ for any $k \geq 20$. It implies that*

$$T^*_\infty = 18.$$

By Inequality (6) in Lemma 3, T_∞ is the smallest t such that

$$1 \geq \frac{9}{2} \left(\frac{8}{9} \right)^{t-1} \Longrightarrow T_\infty = 14.$$

In this example,

$$T^*_\infty = 18 > 14 = T_\infty.$$

This comparison is illustrated in Figure 2.

Proof of Proposition 5. A principal with $m+1$ projects can always dispose one project and guarantee himself the payoff E^{m*}_P. As a result, $E^{m+1*}_P \geq E^{m*}_P$, and hence the sequence $\{E^{m*}_P\}$ is increasing in m. By the same logic, $E^{m*}_P \leq E^{\infty}_P$ for

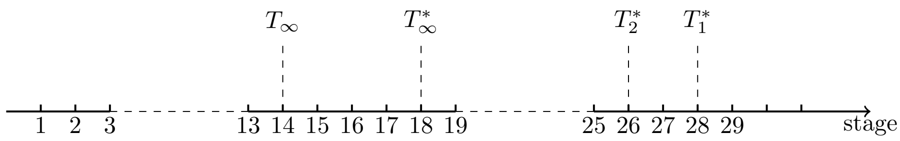

Figure 2: The limit case

any $m \geq 1$. Thus, $\{E_P^{m*}\}$ is convergent. Let $E_P^{\infty*}$ be the limit of the sequence $\{E_P^{m*}\}$. It is clear that $E_P^{\infty*} \leq E_P^{\infty}$.

Since T^*_∞ is the limit of the decreasing sequence $\{T^*_m\}$, we then have that T^*_∞ is the smallest t such that

$$v - c_H \geq \frac{\lambda_L}{\lambda_H}(c_H - c_L)\frac{\left[1 - \delta(\alpha_H - \alpha_L)\right]}{1 - \delta\alpha_H}\left(\frac{\alpha_H - \alpha_L}{\alpha_H}\right)^{t-1}. \tag{7}$$

Since T_∞ is the smallest t such that

$$v - c_H \geq \frac{\lambda_L}{\lambda_H}(c_H - c_L)\left(\frac{\alpha_H - \alpha_L}{\alpha_H}\right)^{t-1},$$

$T^*_\infty \geq T_\infty$ as

$$\frac{\alpha_H - \alpha_L}{\alpha_H} < 1 \text{ and } \frac{1 - \delta(\alpha_H - \alpha_L)}{1 - \delta\alpha_H} \geq 1.$$

□

Remark 6. *In Proposition 5, we show that the principal's equilibrium payoff E_P^{m*} and the optimal mechanism in Proposition 3 may be distinct from the equilibrium payoff E_P^{∞} and the optimal mechanism in the case without the capacity constraint. There are two notable differences.*[16]

- *Suppose that Condition 1 (a) holds. When the capacity constraint is present, the principal never assigns a project at a high cost. As a result, he cannot extract rent from a high-cost agent even though the agent has a low cost at some previous stage. On the other hand, when the capacity constraint is absent, as long as the low cost has appeared before, the principal will assign projects to the agent regardless of her type. Thus, the principal may be able to extract some rent from a high-cost agent.*

[16]The key in the analysis of the case with the capacity constraint is that the principal needs to take into account the opportunity cost. That is, the principal compares the payoff by assigning one project at a high cost, with the payoff by holding and reassigning it in the future. Such consideration does not exist in the case without the capacity constraint.

- *Suppose that Condition 1 (b) holds. Note that the definition of T_∞ in Lemma 3 does not rely on the discount factor δ, while the definition of T^*_∞ does.*[17] *The deadline T^*_∞ is greater than T_∞, and they coincide only when δ is sufficiently small (i.e., $\delta = 0$), implying that the assignments of the projects will be further delayed in the case with the capacity constraint.*

6 Conclusion

In this paper, we address the question of how a principal uses delay as an instrument to optimally assign multiple projects to an agent with changing costs in a dynamic environment. We fully characterize the optimal dynamic mechanism. It has been shown that the key to the construction of the optimal mechanism is an intertemporal trade-off: the principal compares the payoff from assigning the projects right away with the payoff from withholding and reassigning them in the future. This trade-off induces interesting properties of the assignment rule, and shapes the dynamics of the optimal mechanism accordingly. In addition, our results have further implications when there is a large number of projects. We show that the presence of the capacity constraint could delay the assignment of the projects and reduce the principal's payoff. In reality, the principal often has capacity constraints for various (budget/headcount/quota) reasons. This paper contributes to our understanding of these environments.

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7 Appendix

7.1 Proofs of Lemma 1 and Proposition 2

We prove Lemma 1 by modifying the argument in Battaglini (2005).

Proof of Lemma 1.

Given the history c^{t-1} and s^t for $t \geq 1$. Suppose that $s_t = 1$. That is, the project has not been completed at stage t. Recall that $\tilde{s}^{t+1} = (s^t, s_t - 1)$ and $\hat{s}^{t+1} = (s^t, s_t)$ for $t \geq 1$. In order for the agent not to misreport high cost when she has low cost at stage t, we must have

$$\begin{aligned}
&U(c^{t-1}, c_L; s^t) \\
\geq& -q_t(c^{t-1}, c_H; s^t)c_L + p_t(c^{t-1}, c_H; s^t) \\
&+ \delta q_t(c^{t-1}, c_H; s^t)\mathbb{E}\left[U(c^{t-1}, c_H, c_{t+1}; \tilde{s}^{t+1})|c_L\right] \\
&+ \delta(1 - q_t(c^{t-1}, c_H; s^t))\mathbb{E}\left[U(c^{t-1}, c_H, c_{t+1}; \hat{s}^{t+1})|c_L\right]
\end{aligned}$$

$$
\begin{aligned}
&= U(c^{t-1}, c_H; s^t) + (c_H - c_L) q_t(c^{t-1}, c_H; s^t) \\
&\quad + \delta q_t(c^{t-1}, c_H; s^t)(\alpha_H - \alpha_L)[U(c^{t-1}, c_H, c_L; \tilde{s}^{t+1}) - U(c^{t-1}, c_H, c_H; \tilde{s}^{t+1})] \\
&\quad + \delta(1 - q_t(c^{t-1}, c_H; s^t))(\alpha_H - \alpha_L)[U(c^{t-1}, c_H, c_L; \hat{s}^{t+1}) - U(c^{t-1}, c_H, c_H; \hat{s}^{t+1})].
\end{aligned}
$$

Since $\tilde{s}^{t+1} = (s^t, s_t - 1)$ and $s_t - 1 = 0$. We have

$$
U(c^{t-1}, c_H, c_L; \tilde{s}^{t+1}) = U(c^{t-1}, c_H, c_H; \tilde{s}^{t+1}) = 0,
$$

and hence

$$
\begin{aligned}
&U(c^{t-1}, c_L; s^t) - U(c^{t-1}, c_H; s^t) \\
&\geq (c_H - c_L) q_t(c^{t-1}, c_H; s^t) \\
&\quad + \delta(1 - q_t(c^{t-1}, c_H; s^t))(\alpha_H - \alpha_L)[U(c^{t-1}, c_H, c_L; \hat{s}^{t+1}) - U(c^{t-1}, c_H, c_H; \hat{s}^{t+1})].
\end{aligned}
$$

The above inequality gives a lower bound of the additional payoff the agent could enjoy when the cost is low rather than high.

Suppose that for some $i' \geq 0$,

$$
\begin{aligned}
&U(c^{t-1}, c_L; s^t) - U(c^{t-1}, c_H; s^t) \\
&\geq (c_H - c_L) \sum_{i=0}^{i'} \delta^i (\alpha_H - \alpha_L)^i q_{t+i}(c^{t-1}, c_H^{i+1}; s^{t+i}) \prod_{1 \leq j \leq i} [1 - q_{t+j-1}(c^{t-1}, c_H^j; s^{t+j-1})] \\
&\quad + \delta^{i'+1} (\alpha_H - \alpha_L)^{i'+1} \prod_{1 \leq j \leq i'+1} [1 - q_{t+j-1}(c^{t-1}, c_H^j; s^{t+j-1})] \cdot \\
&\quad [U(c^{t-1}, c_H^{i'+1}, c_L; \hat{s}^{t+i'+1}) - U(c^{t-1}, c_H^{i'+1}, c_H; \hat{s}^{t+i'+1})].
\end{aligned}
$$

By the induction argument,

$$
\begin{aligned}
&U(c^{t-1}, c_L; s^t) - U(c^{t-1}, c_H; s^t) \\
&\geq (c_H - c_L) \sum_{i=0}^{i'+1} \delta^i (\alpha_H - \alpha_L)^i q_{t+i}(c^{t-1}, c_H^{i+1}; s^{t+i}) \prod_{1 \leq j \leq i} [1 - q_{t+j-1}(c^{t-1}, c_H^j; s^{t+j-1})] \\
&\quad + \delta^{i'+2} (\alpha_H - \alpha_L)^{i'+2} \prod_{1 \leq j \leq i'+2} [1 - q_{t+j-1}(c^{t-1}, c_H^j; s^{t+j-1})] \cdot \\
&\quad [U(c^{t-1}, c_H^{i'+2}, c_L; \hat{s}^{t+i'+2}) - U(c^{t-1}, c_H^{i'+2}, c_H; \hat{s}^{t+i'+2})].
\end{aligned}
$$

Taking i' to infinity, since $[U(c^{t-1}, c_H^{i'+2}, c_L; \hat{s}^{t+i'+2}) - U(c^{t-1}, c_H^{i'+2}, c_H; \hat{s}^{t+i'+2})]$ is bounded, we have

$$
\begin{aligned}
&U(c^{t-1}, c_L; s^t) - U(c^{t-1}, c_H; s^t) \\
&\geq (c_H - c_L) \sum_{i=0}^{\infty} \delta^i (\alpha_H - \alpha_L)^i q_{t+i}(c^{t-1}, c_H^{i+1}; s^{t+i}) \prod_{1 \leq j \leq i} [1 - q_{t+j-1}(c^{t-1}, c_H^j; s^{t+j-1})].
\end{aligned}
$$

This completes the proof. □

Proof of Proposition 2.

The proof proceeds in two steps. We first show that the assignment rule given in Proposition 2 maximizes E_P^1. Then we construct the payment rule and check that the corresponding mechanism satisfies all incentive constraints and participation constraints.

Step 1. Recall that

$$\begin{aligned}E_P^1 = &\sum_{c_1 \in \{c_H, c_L\}} \lambda(c_1)\Big\{q_1(c_1;s_1)(v-c_1) \\ &+ \delta[1-q_1(c_1;s_1)]\big(W(c_1,c_H;s_1,s_1)P(c_H|c_1) + W(c_1,c_L;s_1,s_1)P(c_L|c_1)\big)\Big\} \\ &- \lambda_L(c_H-c_L)\sum_{i=0}^{\infty}\delta^i(\alpha_H-\alpha_L)^i q_{i+1}(c_H^{i+1};s^{i+1})\prod_{1\le j\le i}[1-q_j(c_H^j;s^j)].\end{aligned}$$

We shall check that the assignment rule given by claims (1-3) in Proposition 2 maximizes E_P^1.

(1) This case is obvious. For any c^{t-1} and s^t such that $s_t = 1$, since

$$v - c_L > \delta(W(c^{t-1}, c_L, c_H; s^{t+1})\alpha_L + W(c^{t-1}, c_L, c_L; s^{t+1})(1-\alpha_L)),$$

we have that $q_t(c^{t-1}, c_L; s^t) = 1$.

(2-3) Suppose that the principal chooses some $t' \geq 1$, and lets $q_t(c_H^t; s^t) = 0$ for any $t < t'$ and $q_{t'}(c_H^{t'}; s^{t'}) = 1$. Note that no assignment will happen after stage t' since the principal assigns the project at stage t' regardless of the report. As a result, we can ignore the surplus in E_P^1 which is from the path with the initial cost being c_L, and focus on the following part:

$$\begin{aligned}&\lambda_H\delta\sum_{k\ge 0}^{t'-2}[\delta^k P(c_H|c_H)^k]P(c_L|c_H)(v-c_L) + \lambda_H\delta^{t'-1}P(c_H|c_H)^{t'-1}(v-c_H) \\ &- \lambda_L(c_H-c_L)\delta^{t'-1}(\alpha_H-\alpha_L)^{t'-1} \\ =\ &\lambda_H\delta(1-\alpha_H)(v-c_L)\frac{1-\delta^{t'-1}\alpha_H^{t'-1}}{1-\delta\alpha_H} + \lambda_H\delta^{t'-1}\alpha_H^{t'-1}(v-c_H) \\ &- \lambda_L(c_H-c_L)\delta^{t'-1}(\alpha_H-\alpha_L)^{t'-1}.\end{aligned} \tag{8}$$

If the principal chooses never to assign the project along the path $(c_H, c_H, \ldots)$ (i.e., $q_t^*(c_H^t; s^t) = 0$ for any t), then the relevant part in the principal's payoff is

$$\lambda_H\delta(1-\alpha_H)(v-c_L)\frac{1}{1-\delta\alpha_H}. \tag{9}$$

The difference of the payoffs for these two choices is

$$(8) - (9) = -\lambda_H\delta(1-\alpha_H)(v-c_L)\frac{\delta^{t'-1}\alpha_H^{t'-1}}{1-\delta\alpha_H} + \lambda_H\delta^{t'-1}\alpha_H^{t'-1}(v-c_H)$$

$$
\begin{aligned}
&- \lambda_L(c_H - c_L)\delta^{t'-1}\left(\alpha_H - \alpha_L\right)^{t'-1} \\
&= \lambda_H\delta^{t'-1}\alpha_H^{t'-1}\Bigg[(v - c_H) - \frac{\delta(1-\alpha_H)}{1-\delta\alpha_H}(v - c_L) \\
&\quad - \frac{\lambda_L}{\lambda_H}(c_H - c_L)\left(\frac{\alpha_H - \alpha_L}{\alpha_H}\right)^{t'-1}\Bigg],
\end{aligned}
$$

which is negative for any t' if

$$
v - c_H \leq \delta\Bigg((v - c_L)(1-\alpha_H) + (v - c_H)\alpha_H\Bigg).
$$

In this case, the principal should choose never to assign the project along the path $(c_H, c_H, \ldots)$. Thus, for any s^t such that $s_t = 1$, $q_t(c_H^t; s^t) = 0$. This gives the assignment rule in Proposition 2 (2).

If

$$
v - c_H > \delta\Bigg((v - c_L)(1-\alpha_H) + (v - c_H)\alpha_H\Bigg),
$$

then

$$
(v - c_H) - \frac{\delta(1-\alpha_H)}{1-\delta\alpha_H}(v - c_L) > \frac{\lambda_L}{\lambda_H}(c_H - c_L)\left(\frac{\alpha_H - \alpha_L}{\alpha_H}\right)^{t'-1}
$$

holds for sufficiently large t'. As a result, the principal should choose to assign the project at some (finite) stage. The question is then reduced to pick a stage t' such that (8) is maximized.

Denote $a = \frac{\lambda_H}{\delta\alpha_H}[(v-c_H) - \frac{\delta(1-\alpha_H)(v-c_L)}{1-\delta\alpha_H}]$, $b = \delta\alpha_H$, $c = \frac{\lambda_L}{\delta(\alpha_H - \alpha_L)}(c_H - c_L)$, $d = \delta(\alpha_H - \alpha_L)$. Define a function as $g(t) = ab^t - cd^t$. Taking the first order derivative of g with respect to t, one gets $g'(t) = (a\ln b)b^t - (c\ln d)d^t$. Then $g'(t) \geq 0$ if and only if $t \leq \ln(\frac{a\ln b}{c\ln d})/\ln\frac{d}{b}$. As a result, g is increasing until $\ln(\frac{a\ln b}{c\ln d})/\ln\frac{d}{b}$ and then decreasing. Notice that (8) $-$ (9) $= ab^{t'} - cd^{t'}$, and the principal's payoff is the summation of $ab^{t'} - cd^{t'}$ and a constant (9). Thus, the principal's payoff is single-peaked: increasing in terms of t' until $\ln(\frac{a\ln b}{c\ln d})/\ln\frac{d}{b}$, and then (strictly) decreasing afterwards. To identify the optimal stage to assign the project along the path $(c_H, c_H, \ldots)$, we need to find the smallest t such that $g(t-1) \geq g(t)$, which implies that

$$
\begin{aligned}
v - c_H \geq\ & \delta\Bigg((v - c_L)(1-\alpha_H) + (v - c_H)\alpha_H\Bigg) \\
&+ \frac{\lambda_L}{\lambda_H}(c_H - c_L)\Big[1 - \delta(\alpha_H - \alpha_L)\Big]\left(\frac{\alpha_H - \alpha_L}{\alpha_H}\right)^{t-1}.
\end{aligned}
$$

This gives the assignment rule in Proposition 2 (3).

Step 2. Next, we construct the payment rule and check that the corresponding mechanism satisfies all the incentive constraints and participation constraints.

Given c^{t-1} and s^t with $s_t = 1$,

$$
U(c^{t-1}, c_L; s^t) =
$$

$$(c_H - c_L)\sum_{i=0}^{\infty}\delta^i(\alpha_H-\alpha_L)^i q_{t+i}(c^{t-1}, c_H^{i+1}; s^{t+i}) \prod_{1\le j\le i}[1-q_{t+j-1}(c^{t-1}, c_H^j; s^{t+j-1})]$$

and

$$U(c^{t-1}, c_H; s^t) = 0.$$

Let

$$p_t(c^{t-1}, c_t; s^t) =$$
$$U(c^{t-1}, c_t; s^t) + q_t(c^{t-1}, c_t; s^t)c_t - \delta(1-q_t(c^{t-1}, c_t; s^t))\mathbb{E}(U(c^{t-1}, c_t, c_{t+1}; \hat{s}^{t+1})|c_t).$$

It is immediate to check that all the participation constraints are satisfied and the agent gets the payoff $U(c^{t-1}, c_t; s^t)$ at the cost c_t given c^{t-1} and s^t. Following the argument in the proof of Lemma 1, the agent with low cost would not report high cost. We only need to check that the agent with high cost would not report low cost. Given c^{t-1} and s^t, if the agent has cost c_H but reports c_L, then her payoff is

$$\begin{aligned}
&-q_t(c^{t-1}, c_L; s^t)c_H + p_t(c^{t-1}, c_L; s^t)\\
&+\delta(1-q_t(c^{t-1}, c_L; s^t))\mathbb{E}\left[U(c^{t-1}, c_L, c_{t+1}; \hat{s}^{t+1})|c_H\right]\\
=\,&U(c^{t-1}, c_L; s^t) - (c_H - c_L)\\
\le\,&0\\
=\,&U(c^{t-1}, c_H; s^t).
\end{aligned}$$

The first equality holds since $q_t(c^{t-1}, c_L; s^t) = 1$. The inequality is true since $U(c^{t-1}, c_L; s^t) \le c_H - c_L$ by definition. This completes the proof. □

7.2 Proof of Proposition 3

The following lemma generalizes Lemmas 1 and 2 to the general environment. It provides a lower bound for the additional payoff of the agent when the cost is low rather than high, and also an upper bound for the principal's payoff.

Lemma 4. *Fix an incentive compatible and individually rational mechanism* Γ *with the assignment rule* $\mathbf{q}^*$.

1. *Given the history* c^{t-1} *and* s^t *with* $s_t = m \ge 1$ *at stage* $t \ge 1$, *we have*

$$\begin{aligned}
&U(c^{t-1}, c_L; s^t) - U(c^{t-1}, c_H; s^t) \qquad (10)\\
\ge\,&(c_H - c_L)\sum_{i_1=0}^{\infty}\Big\{\delta^{i_1}(\alpha_H-\alpha_L)^{i_1} q^*_{t+i_1}(c^{t-1}, c_H^{i_1+1}; s^{t+i_1})\cdot\\
&\prod_{1\le j\le i_1}\left[1-q^*_{t+j-1}(c^{t-1}, c_H^j; s^{t+j-1})\right]\Big\}\\
&+(c_H-c_L)\sum_{\substack{i_1\ge 0\\ i_2\ge 1}}\Big\{\delta^{i_1+i_2}(\alpha_H-\alpha_L)^{i_1+i_2} q^*_{t+i_1}(c^{t-1}, c_H^{i_1+1}; s^{t+i_1})\\
&\cdot q^*_{t+i_1+i_2}(c^{t-1}, c_H^{i_1+i_2+1}; s^{t+i_1+i_2}) \prod_{\substack{1\le j\le i_1+i_2\\ j\ne i_1+1}}\left[1-q^*_{t+j-1}(c^{t-1}, c_H^j; s^{t+j-1})\right]\Big\}
\end{aligned}$$

$$+\cdots$$

$$+ (c_H - c_L) \sum_{\substack{i_1 \geq 0 \\ i_2 \geq 1 \\ \cdots \\ i_m \geq 1}} \Bigg\{ \delta^{\sum_{k=1}^m i_k} (\alpha_H - \alpha_L)^{\sum_{k=1}^m i_k} \prod_{k=1}^{m} q^*_{t+\sum_{l=1}^k i_l}(c^{t-1}, c_H^{\sum_{l=1}^k i_l + 1}; s^{t+\sum_{l=1}^k i_l})$$

$$\cdot \prod_{\substack{1 \leq j \leq \sum_{k=1}^m i_k \\ j \neq i_1 + 1 \\ j \neq i_1 + i_2 + 1 \\ \cdots \\ j \neq \sum_{k=1}^{m-1} i_k + 1}} [1 - q^*_{t+j-1}(c^{t-1}, c_H^j; s^{t+j-1})] \Bigg\}.$$

Let $E_A^m(c^{t-1}, s^t)$ be the summation on the right hand side of the above inequality.

2. *The expected payoff of the principal is no greater than*

$$\begin{aligned} E_P^m = \sum_{c_1 \in \{c_H, c_L\}} \lambda(c_1) \Big\{ & q_1^*(c_1; s_1)(v - c_1) \\ & + \delta q_1^*(c_1; s_1) \left(W(c_1, c_H; s_1, s_1 - 1) P(c_H | c_1) + W(c_1, c_L; s_1, s_1 - 1) P(c_L | c_1) \right) \\ & + \delta [1 - q_1^*(c_1; s_1)] \big(W(c_1, c_H; s_1, s_1) P(c_H | c_1) + W(c_1, c_L; s_1, s_1) P(c_L | c_1) \big) \Big\} \\ & - \lambda_L \cdot E_A^m(c^0, s^1). \end{aligned}$$

Remark 7. *Though the term $E_A^m(c^{t-1}, s^t)$ in Lemma 4 (1) seems complicated at the first glance, it has a very nice structure for interpretation. It is shown in the proof of Proposition 3 below that in the optimal mechanism, the agent gets zero payoff at high cost and $E_A^m(c^{t-1}, s^t)$ at low cost. In addition, Inequality (10) is binding, and the payoff of the principal is E_P^m. For $1 \leq k \leq m$, the k-th summation in $E_A^m(c^{t-1}, s^t)$ corresponds to the expected rents pinned down to the path $(c^{t-1}, c_H, c_H, \ldots)$ and the k-th project. In particular, every such summation in $E_A^m(c^{t-1}, s^t)$ represents the marginal payoff of the agent due to the corresponding project. In the k-th summation, we use the notations $i_1, i_2, \ldots, i_k$ to describe that the first project is assigned i_1 stages later, the second project is assigned $i_1 + i_2$ stages later, so on and so forth. Then each term $\delta^{\sum_{k'=1}^k i_{k'}} (\alpha_H - \alpha_L)^{\sum_{k'=1}^k i_{k'}} (c_H - c_L)$ in the k-th summation is the expected rents of having low cost rather than high cost at stage t, if the k-th project is assigned exactly $\sum_{k'=1}^k i_{k'}$ stages later. The k-th summation is hence simply the aggregate of the expected rents from every such stage.*

The proof of Lemma 4 (1) involves two levels of induction. In the first level, the argument is based on the induction on m, the number of projects. Lemma 1 shows that the claim is true when there is one project. We shall assume that the claim is true in the case with m' projects and then show that it is still true with $m' + 1$ projects. To complete the argument in the case with $m' + 1$ projects, we need another induction argument to deal with the incentive of the low-cost agent and simplify the expression of the lower bound. The argument of the second-level induction is similar to that in the proof of Lemma 1.

Proof of Lemma 4.

(1) We adopt an induction argument. By Lemma 1, the claim would be true if there is only one project left. Suppose that the claim holds in the case with $m'-1$ projects. We consider the case with m' projects. Given the history c^{t-1} and s^t for $t \geq 1$. Suppose that $s_t = m'$. Recall that $\tilde{s}^{t+1} = (s^t, s_t - 1)$ and $\hat{s}^{t+1} = (s^t, s_t)$. In order for the agent not to misreport high cost when she has a low cost at stage t, we have

$$\begin{aligned}
&U(c^{t-1}, c_L; s^t) \\
\geq& -q_t(c^{t-1}, c_H; s^t)c_L + p_t(c^{t-1}, c_H; s^t) \\
&+ \delta q_t(c^{t-1}, c_H; s^t)\mathbb{E}\left[U(c^{t-1}, c_H, c_{t+1}; \tilde{s}^{t+1})|c_L\right] \\
&+ \delta(1 - q_t(c^{t-1}, c_H; s^t))\mathbb{E}\left[U(c^{t-1}, c_H, c_{t+1}; \hat{s}^{t+1})|c_L\right] \\
=& U(c^{t-1}, c_H; s^t) + (c_H - c_L)q_t(c^{t-1}, c_H; s^t) \\
&+ \delta q_t(c^{t-1}, c_H; s^t)(\alpha_H - \alpha_L)[U(c^{t-1}, c_H, c_L; \tilde{s}^{t+1}) - U(c^{t-1}, c_H, c_H; \tilde{s}^{t+1})] \\
&+ \delta(1 - q_t(c^{t-1}, c_H; s^t))(\alpha_H - \alpha_L)[U(c^{t-1}, c_H, c_L; \hat{s}^{t+1}) - U(c^{t-1}, c_H, c_H; \hat{s}^{t+1})].
\end{aligned}$$

That is,

$$\begin{aligned}
&U(c^{t-1}, c_L; s^t) - U(c^{t-1}, c_H; s^t) \\
\geq& (c_H - c_L)q_t(c^{t-1}, c_H; s^t) \\
&+ \delta q_t(c^{t-1}, c_H; s^t)(\alpha_H - \alpha_L)[U(c^{t-1}, c_H, c_L; \tilde{s}^{t+1}) - U(c^{t-1}, c_H, c_H; \tilde{s}^{t+1})] \\
&+ \delta(1 - q_t(c^{t-1}, c_H; s^t))(\alpha_H - \alpha_L)[U(c^{t-1}, c_H, c_L; \hat{s}^{t+1}) - U(c^{t-1}, c_H, c_H; \hat{s}^{t+1})].
\end{aligned}$$

Suppose that for some $i' \geq 0$,

$$\begin{aligned}
&U(c^{t-1}, c_L; s^t) - U(c^{t-1}, c_H; s^t) \\
\geq& (c_H - c_L)\sum_{i=0}^{i'} \delta^i(\alpha_H - \alpha_L)^i q_{t+i}(c^{t-1}, c_H^{i+1}; s^{t+i}) \prod_{1\leq j\leq i}[1 - q_{t+j-1}(c^{t-1}, c_H^j; s^{t+j-1})] \\
&+ \sum_{i=0}^{i'} \Big\{ \delta^{i+1}(\alpha_H - \alpha_L)^{i+1} q_{t+i}(c^{t-1}, c_H^{i+1}; s^{t+i}) \prod_{1\leq j\leq i}[1 - q_{t+j-1}(c^{t-1}, c_H^j; s^{t+j-1})]\cdot \\
&[U(c^{t-1}, c_H^{i+1}, c_L; s^{t+i+1}) - U(c^{t-1}, c_H^{i+1}, c_H; s^{t+i+1})] \Big\} \\
&+ \delta^{i'+1}(\alpha_H - \alpha_L)^{i'+1} \prod_{1\leq j\leq i'+1}[1 - q_{t+j-1}(c^{t-1}, c_H^j; s^{t+j-1})]\cdot \\
&[U(c^{t-1}, c_H^{i'+1}, c_L; \hat{s}^{t+i'+1}) - U(c^{t-1}, c_H^{i'+1}, c_H; \hat{s}^{t+i'+1})].
\end{aligned}$$

Applying the induction argument to $i'+1$, and then take i' to the positive infinity, we have

$$\begin{aligned}
&U(c^{t-1}, c_L; s^t) - U(c^{t-1}, c_H; s^t) \\
\geq& (c_H - c_L)\sum_{i=0}^{\infty} \delta^i(\alpha_H - \alpha_L)^i q_{t+i}(c^{t-1}, c_H^{i+1}; s^{t+i}) \prod_{1\leq j\leq i}[1 - q_{t+j-1}(c^{t-1}, c_H^j; s^{t+j-1})]
\end{aligned}$$

$$+\sum_{i=0}^{\infty}\Big\{\delta^{i+1}(\alpha_H-\alpha_L)^{i+1}q_{t+i}(c^{t-1},c_H^{i+1};s^{t+i})\prod_{1\le j\le i}[1-q_{t+j-1}(c^{t-1},c_H^{j};s^{t+j-1})]\cdot$$
$$[U(c^{t-1},c_H^{i+1},c_L;s^{t+i+1})-U(c^{t-1},c_H^{i+1},c_H;s^{t+i+1})]\Big\}.$$

Recall that we have assumed Inequality (10) to hold in the case with $m'-1$ projects. In the second summation of the right hand side of the above inequality, one (and only one) project has been assigned with probability $q_{t+i}(c^{t-1},c_H^{i+1};s^{t+i})$. Thus, the tracking number s_{t+i+1} in this summation is $m'-1$. Applying Inequality (10) to $U(c^{t-1},c_H^{i+1},c_L;s^{t+i+1})-U(c^{t-1},c_H^{i+1},c_H;s^{t+i+1})$, we then get Inequality (10) for the case with m' projects. This completes the induction argument.

(2) Note that $U(c_H;s^1)\geq 0$ by the participation constraint. Following the same argument as in Section 3, $E_A^m(c^0,s^1)$ is the lower bound of the agent's payoff when the cost is low. Then $\lambda_L\cdot E_A^m(c^0,s^1)$ is the lower bound of the agent's expected rents. The payoff of the principal must be greater than the expected surplus less the lower bound of the agent's expected rents. □

Proof of Proposition 3.

We shall proceed in two steps. The first step is to find the assignment rules which maximize E_P^m. Then a payment rule is constructed so that the corresponding mechanism satisfies all the incentive constraints and participation constraints. Though the second step is standard, the first step here is more difficult, as the assignment of a project may influence the assignment path of the remaining projects.

Step 1. We shall check that the assignment rule given in Proposition 3 maximizes E_P^m.

(1) Suppose that at stage t there are k projects left, and $c_t=c_L$. No matter whether one project is assigned at this stage or not, the principal can always adopt the same strategy for the next $k-1$ projects.

1. If one project is assigned at stage t, then the $k-1$ projects are all the projects left.
2. If no project is assigned at this stage, then the $k-1$ projects are the k-th-to-last project to the second-to-last project.

If the principal adopts the same strategy in these two situations, then the surplus contributed by these $k-1$ projects are the same. Since

$$v-c_L>\delta\Big((v-c_L)(1-\alpha_L)+(v-c_H)\alpha_L\Big),$$

the surplus contributed by an assigned project at stage t (*i.e.*, case (1)) is strictly higher than the surplus contributed by the last project (*i.e.*, case (2)). As a result, the principal shall assign one project as long as the the cost is low.

(2) Suppose that

$$v - c_H \leq \delta\Big((v - c_L)(1 - \alpha_H) + (v - c_H)\alpha_H\Big).$$

We need to distinguish two cases: (a) $c^{t-1} \neq c_H^{t-1}$; (b) $c^{t-1} = c_H^{t-1}$.

(2.a) We first work with the case that $c^{t-1} \neq c_H^{t-1}$. Consider the case that there is only one project left at stage t. Suppose that the principal chooses some $t' \geq 1$, and lets $q_{t+\hat{t}-1}(c^{t-1}, c_H^{\hat{t}}; s^{t+\hat{t}-1}) = 0$ for any $1 \leq \hat{t} < t'$ and $q_{t+t'-1}(c^{t-1}, c_H^{t'}; s^{t+t'-1}) = 1$. The part of surplus in E_P^m from this choice would be

$$\begin{aligned} &\delta \sum_{k\geq 0}^{t'-2} [\delta^k P(c_H|c_H)^k] P(c_L|c_H)(v - c_L) + \delta^{t'-1} P(c_H|c_H)^{t'-1}(v - c_H) \\ = &\delta(1 - \alpha_H)(v - c_L)\frac{1 - \delta^{t'-1}\alpha_H^{t'-1}}{1 - \delta\alpha_H} + \delta^{t'-1}\alpha_H^{t'-1}(v - c_H). \end{aligned}$$

If the principal chooses never to assign the project along the path $(c^{t-1}, c_H, c_H, \ldots)$, then the relevant part in E_P^m is

$$\delta(1 - \alpha_H)(v - c_L)\frac{1}{1 - \delta\alpha_H}.$$

If Condition 1 (a) holds, then the latter is higher than the former for any t', which implies that the principal should choose never to assign the last project given a report c_H.

Suppose that the principal decides that he shall never assign the last $r \geq 1$ projects given a report c_H. Let $\phi_r(c)$ be the expected surplus when there are r projects, the current cost is c, and the principal only assigns the project at c_L. Then

$$\phi_r(c_H) = \delta P(c_L|c_H)\phi_r(c_L) + \delta P(c_H|c_H)\phi_r(c_H),$$

which implies that

$$\phi_r(c_H) = \frac{\delta(1 - \alpha_H)}{1 - \delta\alpha_H}\phi_r(c_L).$$

Consider the case that there are $r + 1$ projects left at stage t. Suppose that the principal chooses some $t' \geq 1$, and lets $q_{t+\hat{t}-1}(c^{t-1}, c_H^{\hat{t}}; s^{t+\hat{t}-1}) = 0$ for any $1 \leq \hat{t} < t'$ and $q_{t+t'-1}(c^{t-1}, c_H^{t'}; s^{t+t'-1}) = 1$, when there are $r + 1$ projects. The part of surplus in E_P^m from this choice is

$$\begin{aligned} &\delta \sum_{k\geq 0}^{t'-2} [\delta^k P(c_H|c_H)^k] P(c_L|c_H)[(v - c_L) + \delta(1 - \alpha_L)\phi_r(c_L) + \delta\alpha_L\phi_r(c_H)] \\ &+ \delta^{t'-1} P(c_H|c_H)^{t'-1}[(v - c_H) + \delta(1 - \alpha_H)\phi_r(c_L) + \delta\alpha_H\phi_r(c_H)]. \end{aligned}$$

If the principal chooses never to assign the project along the path $(c^{t-1}, c_H, c_H, \ldots)$, then the relevant part in E_P^m is

$$\delta(1-\alpha_H)\frac{1}{1-\delta\alpha_H}[(v-c_L)+\delta(1-\alpha_L)\phi_r(c_L)+\delta\alpha_L\phi_r(c_H)].$$

By substituting $\phi_r(c_H) = \frac{\delta(1-\alpha_H)}{1-\delta\alpha_H}\phi_r(c_L)$, it is straightforward to check if Condition 1 (a) holds, then the latter is higher than the former for any t', which implies that the principal should choose never to assign the $r+1$-th to last project given a report c_H. This completes the induction argument.

(2.b) Suppose that $c^{t-1} = c_H^{t-1}$. The proof of this part follows a similar induction argument as that in (2.a). The argument for the one-project case is the same as that in the proof of Proposition 2 (2). Suppose that the principal decides that he shall never assign the last $r \geq 1$ projects given a report c_H. Again, consider the case that there are $r+1$ projects left at stage t. Suppose that the principal chooses some $t' \geq 1$, and lets $q_{t+\hat{t}-1}(c_H^{t-1}, c_H^{\hat{t}}; s^{t+\hat{t}-1}) = 0$ for any $1 \leq \hat{t} < t'$ and $q_{t+t'-1}(c_H^{t-1}, c_H^{t'}; s^{t+t'-1}) = 1$, when there are $r+1$ projects. The surplus in E_P^m from this choice would be

$$\begin{aligned}
&\delta\lambda_H \sum_{k\geq 0}^{t'-2}[\delta^k P(c_H|c_H)^k]P(c_L|c_H)[(v-c_L)+\delta(1-\alpha_L)\phi_r(c_L)+\delta\alpha_L\phi_r(c_H)]\\
&+\lambda_H\delta^{t'-1}P(c_H|c_H)^{t'-1}[(v-c_H)+\delta(1-\alpha_H)\phi_r(c_L)+\delta\alpha_H\phi_r(c_H)]\\
&-\lambda_L(c_H-c_L)\delta^{t+t'-1}\left(\alpha_H-\alpha_L\right)^{t+t'-1}.
\end{aligned}$$

If the principal chooses never to assign the project along the path $(c_H^{t-1}, c_H, c_H, \ldots)$, then the relevant part in E_P^m is

$$\delta(1-\alpha_H)\frac{1}{1-\delta\alpha_H}[(v-c_L)+\delta(1-\alpha_L)\phi_r(c_L)+\delta\alpha_L\phi_r(c_H)].$$

Once again, if Condition 1 (a) holds, then the latter is higher than the former for any t', which implies that the principal should choose never to assign the $r+1$-th to last project given a report c_H. This completes the induction argument.

(3) Suppose that Condition 1 (b) holds.

(3.a) We prove the claim by induction. Suppose that there is only one project left; that is, $s_t = 1$. If the project is assigned, then the surplus contributed by this project is $v - c_H$. Otherwise, since this is the last project, the principal can get at most $\delta\big((v-c_L)(1-\alpha_H)+(v-c_H)\alpha_H\big)$. Because of the condition, $q_t^*(c^{t-1}, c_H; s^t) = 1$.

Suppose that $q_{t'}^*(c^{t'}; s^{t'}) = 1$ for any $c^{t'} \neq c_H^{t'}$ such that $1 \leq s_{t'} \leq k-1$. Now we consider the case that there are $k \geq 1$ projects left ($s_t = k$). We assume that the principal chooses some smallest integer $j \geq 1$ for $c^{t-1} \neq c_H^{t-1}$ such that $s_t = k$ and $q_{t+j-1}^*(c^{t-1}, c_H^j; s^{t+j-1}) = 1$. Let $W^j(c^{t-1}, c_H; s^t)$ be the future expected surplus at stage t for this particular choice of j. Then

$$W^j(c^{t-1}, c_H; s^t) - W^{j+1}(c^{t-1}, c_H; s^t) = \delta^{j-1}\alpha_H^{j-1}\big[W^1(c^{t-1}, c_H; s^t) - W^2(c^{t-1}, c_H; s^t)\big].$$

In particular,

$$W^1(c^{t-1}, c_H; s^t) = v - c_H + \delta \sum_{c_{t+1}} (v - c_{t+1}) P(c_{t+1}|c_H) + \cdots$$
$$+ \delta^{k-1} \sum_{c_{t+1},\cdots,c_{t+k-1}} (v - c_{t+k-1}) P(c_{t+1}|c_H) \prod_{2 \le i \le k-1} P(c_{t+i}|c_{t+i-1}),$$

and

$$W^2(c^{t-1}, c_H; s^t) = \delta \sum_{c_{t+1}} (v - c_{t+1}) P(c_{t+1}|c_H) + \cdots$$
$$+ \delta^{k} \sum_{c_{t+1},\cdots,c_{t+k}} (v - c_{t+k}) P(c_{t+1}|c_H) \prod_{2 \le i \le k} P(c_{t+i}|c_{t+i-1}).$$

In the calculation of $W^1(c^{t-1}, c_H; s^t)$ and $W^2(c^{t-1}, c_H; s^t)$, we have used the assumption that $q^*_{t'}(c^{t'}; s^{t'}) = 1$ for any $c^{t'} \neq c^{t'}_H$ such that $1 \le s_{t'} \le k-1$. To understand $W^1(c^{t-1}, c_H; s^t)$, the first term is the surplus at stage t since one project is assigned at this stage. Then there are $k-1$ projects left. Based on the induction hypothesis, the principal shall continuously assign one project to the agent at each subsequent stage. The term $W^2(c^{t-1}, c_H; s^t)$ can be understood similarly.

Then we have

$$\begin{aligned} & W^1(c^{t-1}, c_H; s^t) - W^2(c^{t-1}, c_H; s^t) \\ = {} & v - c_H - \delta^k \sum_{c_{t+1},\cdots,c_{t+k}} (v - c_{t+k}) P(c_{t+1}|c_H) \prod_{2 \le i \le k} P(c_{t+i}|c_{t+i-1}). \end{aligned}$$

Using the inequality $v - c_H > \delta\big((v - c_L)(1 - \alpha_H) + (v - c_H)\alpha_H\big)$ iteratively, we have

$$v - c_H \ge \delta^k \sum_{c_{t+1},\cdots,c_{t+k}} (v - c_{t+k}) P(c_{t+1}|c_H) \prod_{2 \le i \le k} P(c_{t+i}|c_{t+i-1}).$$

which implies that $W^1(c^{t-1}, c_H; s^t) - W^2(c^{t-1}, c_H; s^t) > 0$. As a result, for any $j \ge 1$,

$$W^j(c^{t-1}, c_H; s^t) - W^{j+1}(c^{t-1}, c_H; s^t) > 0.$$

That is, $\{W^j(c^{t-1}, c_H; s^t)\}$ is a decreasing sequence in terms of j. If the principal chooses $q^*_{t+j-1}(c^{t-1}, c^j_H; s^{t+j-1}) = 0$ for any $s_{t+j-1} = k$ and $j \ge 1$, then the future expected surplus is denoted by $W^\infty(c^{t-1}, c_H; s^t)$. It is obvious that $W^\infty(c^{t-1}, c_H; s^t)$ is the limit of the decreasing sequence $\{W^j(c^{t-1}, c_H; s^t)\}$, and hence is less than $W^1(c^{t-1}, c_H; s^t)$. As a result, the principal should choose to assign the project immediately: $q^*_t(c^{t-1}, c_H; s^t) = 1$. The proof of this claim is thus completed.

(3.b) Again, we prove the claim by induction. In particular, when considering the principal's payoff, we can ignore the surplus induced along the path with the initial cost being c_L.

We first consider the case that there is only one project last. Suppose that the first to the second-to-last units are assigned along the path $(c_H, c_H, \dots)$ at stages $t_m < t_{m-1} < \dots < t_2$, respectively. That is, if the cost is always high, then the principal shall assign the first project at stage t_m, the second one at stage t_{m-1}, ..., and the second to last project at stage t_2. Note that we adopt the reverse order here to describe the assignment order.

If the principal chooses $q_t^*(c_H^t; s^t) = 0$ for any $t > t_2$, then the surplus of E_P^m from this project is

$$\lambda_H \delta(v - c_L)(1 - \alpha_H) \sum_{k \geq t_2 - 1} \delta^k \alpha_H^k.$$

Suppose that the principal chooses some $T > t_2$ and lets $q_t^*(c_H^t; s^t) = 0$ for $t_2 < t < T$ and $q_T^*(c_H^T; s^T) = 1$. The surplus of E_P^m from this project is

$$\begin{aligned}&\lambda_H \delta(v - c_L)(1 - \alpha_H) \sum_{k \geq t_2 - 1}^{T-2} \delta^k \alpha_H^k + \lambda_H \delta^{T-1} \alpha_H^{T-1}(v - c_H) \\ &- \lambda_L (c_H - c_L) \delta^{T-1} (\alpha_H - \alpha_L)^{T-1}.\end{aligned}$$

The latter is higher than the former if

$$v - c_H \geq \frac{\delta(v - c_L)(1 - \alpha_H)}{1 - \delta \alpha_H} + \frac{\lambda_L}{\lambda_H}(c_H - c_L) \left(\frac{\alpha_H - \alpha_L}{\alpha_H} \right)^{T-1},$$

which holds for sufficiently large T due to the condition that $v - c_H > \delta\big((v - c_L)(1 - \alpha_H) + (v - c_H)\alpha_H\big)$. Following the same argument as that in the proof of Proposition 2 (3), the deadline T_1^{m*} of the last project is the same as that in the case $m = 1$.

In summary, if $T_1^{m*} > t_2$, then the principal will assign the last project to the agent before the deadline T_1^{m*} only when the report is c_L after a sequence of high costs, and assign the project regardless of the report at stage T_1^{m*}. If $T_1^{m*} \leq t_2$, the principal shall assign the project to the agent right after stage t_2.

Suppose that the claim is true for $r - 1$, and $T_1^{m*} \geq T_2^{m*} \geq \dots \geq T_{r-1}^{m*}$, where T_j^{m*} is the deadline to assign the j-th to last project along the path $(c_H, c_H, \dots)$ for $1 \leq j \leq r - 1$. Now we consider the assignment rule of the r-th-to-last project. We take $t_m, \dots, t_{r+1}$ as given.[18] The aim is to identify the deadline to assign the r-th-to-last project. We assume that $t_{r+1} < T_{r-1}^{m*} < \dots < T_1^{m*}$ for simplicity.[19] There are three possibilities: the principal assigns the project before the next deadline T_{r-1}^{m*}, or after that deadline, or never.

I. If the principal decides to assign the r-th-to-last project at some stage i with $t_{r+1} < i < T_{r-1}^{m*}$ along the path $(c_H, c_H, \dots)$, then the relevant surplus in E_P^m

[18] As in the previous case, if the cost is always high, then the principal shall assign the first project at stage t_m, the second one at stage t_{m-1}, ..., and the $r + 1$-th to last project at stage t_{r+1}.

[19] This is without loss of generality. Otherwise, the analysis below is the same by taking $T_{r-1}^{m*} = t_{r+1} + 1$, and $T_j^{m*} = T_{j+1}^{m*} + 1$ for $1 \leq j \leq r - 2$.

after stage t_{r+1} is

$$
\begin{aligned}
&\lambda_H \delta(1-\alpha_H) \sum_{t_{r+1}-1 \le k \le i-2} \delta^k \alpha_H^k \left[f_1(c_L) + \cdots + f_r(c_L)\right] \\
&+ \lambda_H \delta^{i-1} \alpha_H^{i-1} (v - c_H) \\
&+ \lambda_H \delta(1-\alpha_H) \sum_{i-1 \le k \le T_{r-1}^{m*}-2} \delta^k \alpha_H^k \left[f_1(c_L) + \cdots + f_{r-1}(c_L)\right] \\
&+ \lambda_H \delta^{T_{r-1}^{m*}-1} \alpha_H^{T_{r-1}^{m*}-1} (v - c_H) \\
&+ \cdots \\
&+ \lambda_H \delta(1-\alpha_H) \sum_{T_2^{m*}-1 \le k \le T_1^{m*}-2} \delta^k \alpha_H^k f_1(c_L) \\
&+ \lambda_H \delta^{T_1^{m*}-1} \alpha_H^{T_1^{m*}-1} (v - c_H) \\
&- \lambda_L (c_H - c_L) \left[\delta^{i-1} \left(\alpha_H - \alpha_L\right)^{i-1} + \sum_{1 \le k \le r-1} \delta^{T_k^{m*}-1} \left(\alpha_H - \alpha_L\right)^{T_k^{m*}-1} \right].
\end{aligned}
$$

The term $\lambda_H \delta(1-\alpha_H)\delta^k \alpha_H^k \left[f_1(c_L) + \cdots + f_r(c_L)\right]$ represents the expected surplus in the case that the r-th-to-last project is assigned upon a report c_L before stage i, and then the rest is assigned continuously at the subsequent stages. The term $\lambda_H \delta^{i-1}\alpha_H^{i-1}(v-c_H)$ is the expected payoff contributed by the r-th-to-last project in the case that the reports after stage t_{r+1} are always c_H until stage i. The term $\lambda_H \delta(1-\alpha_H)\delta^k \alpha_H^k \left[f_1(c_L) + \cdots + f_{r-1}(c_L)\right]$ describes the payoff in the case that the $r-1$-th-to-last project is assigned upon a report c_L after stage i, but before stage T_{r-1}^{m*}, and then the rest is assigned continuously at the subsequent stages. The term $\lambda_H \delta^{T_{r-1}^{m*}-1} \alpha_H^{T_{r-1}^{m*}-1}(v - c_H)$ is the expected payoff contributed by the $r-1$-th-to-last project in the case that the report is always c_H until the deadline T_{r-1}^{m*}. The other terms can be explained similarly. The last term is the rent left to the agent.

II. Suppose that the principal assigns the r-th-to-last project at stage $i \ge T_{r-1}^{m*}$. Let $T_0^{m*} = \infty$, and

$$
I_j = \{i \colon i \ge T_{r-1}^{m*}, j \text{ is the largest } j' \text{ such that } j' \le r-2, T_{j'}^{m*} + j' > i + r - 1\}
$$

for $0 \le j \le r-2$. If $i \in I_j$, then $i + 1 > T_{r-1}^{m*}$, and hence the $r-1$-th-to-last project will be assigned at stage $i+1$ immediately. The interpretation of the subscript j is that it is the first number that the j-th-to-last project could be assigned no later than its deadline. That is, $T_j^{m*} > i + (r-1-j)$, where i is the stage at which the r-th-to-last project is assigned, and $r-1-j$ is the total number of projects that are assigned after their deadlines because $i \ge T_{r-1}^{m*}$. Then $\{I_j\}_{0 \le j \le r-2}$ is a partition of the sequence $\{T_{r-1}^{m*}, T_{r-1}^{m*}+1, \ldots\}$ such that $i > i'$ for any $i \in I_j$ and $i' \in I_{j'}$ with $j < j'$.

If $i \in I_j$, then the payoff of the principal after stage t_{r+1} is

$$\begin{aligned}\Pi^r(j,i) = {} & \lambda_H \delta(1-\alpha_H) \sum_{t_{r+1}-1 \le k \le i-2} \delta^k \alpha_H^k \left[f_1(c_L) + \cdots + f_r(c_L) \right] \\ & + \lambda_H \delta^{i-1} \alpha_H^{i-1}(v - c_H) \\ & + \lambda_H \delta(1-\alpha_H) \delta^{i-1} \alpha_H^{i-1} \left[f_1(c_L) + \cdots + f_{r-1}(c_L) \right] \\ & + \lambda_H \delta^i \alpha_H^i (v - c_H) \\ & + \cdots \\ & + \lambda_H \delta(1-\alpha_H) \delta^{i+r-j-3} \alpha_H^{i+r-j-3} \left[f_1(c_L) + \cdots + f_{j+1}(c_L) \right] \\ & + \lambda_H \delta^{i+r-j-2} \alpha_H^{i+r-j-2} (v - c_H) \\ & + \lambda_H \delta(1-\alpha_H) \sum_{i+r-j-2 \le k \le T_j^{m*}-2} \delta^k \alpha_H^k \left[f_1(c_L) + \cdots + f_j(c_L) \right] \\ & + \lambda_H \delta^{T_j^{m*}-1} \alpha_H^{T_j^{m*}-1} (v - c_H) \\ & + \cdots \\ & + \lambda_H \delta(1-\alpha_H) \sum_{T_2^{m*}-1 \le k \le T_1^{m*}-2} \delta^k \alpha_H^k (v - c_H) \\ & + \lambda_H \delta^{T_1^{m*}-1} \alpha_H^{T_1^{m*}-1} (v - c_H) \\ & - \lambda_L (c_H - c_L) \Big[\sum_{0 \le k \le r-j-1} \delta^{i+k-1} (\alpha_H - \alpha_L)^{i+k-1} \\ & + \sum_{1 \le k \le j} \delta^{T_k^{m*}-1} (\alpha_H - \alpha_L)^{T_k^{m*}-1} \Big].\end{aligned}$$

The term $\lambda_H \delta(1-\alpha_H)\delta^k \alpha_H^k \left[f_1(c_L) + \cdots + f_r(c_L)\right]$ represents the expected payoff in the case that the r-th-to-last project is assigned upon a report c_L before stage i, and then the rest is assigned continuously at the subsequent stages. The term $\lambda_H \delta^{i-1}\alpha_H^{i-1}(v - c_H)$ is the expected payoff contributed by the r-th-to-last project in the case that the report after stage t_{r+1} is always c_H until stage i. The term $\lambda_H \delta(1-\alpha_H)\delta^{i-1}\alpha_H^{i-1}\left[f_1(c_L) + \cdots + f_{r-1}(c_L)\right]$ describes the expected payoff in the case that the r-th-to-last project is assigned until stage i along the path $(c_H, c_H, \ldots)$ and then the report at the next stage is c_L. The other terms can be explained similarly. The last term is the rent left to the agent.

III. If the principal never assigns the r-th-to-last project along the path $(c_H, c_H, \ldots)$, then he gets the following expected payoff after stage t_{r+1},

$$\lambda_H \delta(1-\alpha_H) \sum_{k \ge t_{r+1}-1} \delta^k \alpha_H^k \left[f_1(c_L) + \cdots + f_r(c_L) \right].$$

The payoff in the case (III) is the limit of the payoff in the case (II) by taking i to the positive infinity.

We first consider case (I) above. To simplify the notation, we collect all the terms which do not depend on i in the payoff of case (I), and denote it by C_1.[20] The payoff in case (I) can be rewritten as

$$\begin{aligned}
&C_1 - \lambda_H \delta(1-\alpha_H) f_r(c_L) \frac{\delta^{i-1}\alpha_H^{i-1}}{1-\delta\alpha_H} + \lambda_H \delta^{i-1}\alpha_H^{i-1}(v-c_H) \\
&- \lambda_L (c_H - c_L)\delta^{i-1}(\alpha_H - \alpha_L)^{i-1} \\
= &C_1 + \lambda_H \delta^{i-1}\alpha_H^{i-1}\big[(v-c_H) - \frac{\delta(1-\alpha_H)f_r(c_L)}{1-\delta\alpha_H} \\
&- \frac{\lambda_L}{\lambda_H}(c_H - c_L)\left(\frac{\alpha_H-\alpha_L}{\alpha_H}\right)^{i-1}\big].
\end{aligned}$$

Following the same argument as in the proof of Proposition 2, the payoff in the above equation is increasing in i up to some point and then decreasing. In particular, the optimal stage $\tilde{t}_r$ to assign this project is the smallest t such that

$$\begin{aligned}
(v-c_H) \geq \delta\left((v-c_H)\alpha_H + f_r(c_L)(1-\alpha_H)\right) \\
+ \frac{\lambda_L}{\lambda_H}(c_H-c_L)\left[1-\delta(\alpha_H-\alpha_L)\right]\left[\frac{\alpha_H-\alpha_L}{\alpha_H}\right]^{t-1}.
\end{aligned}$$

By Proposition 4, $f_r(c_L) \leq f_{r-1}(c_L)$, and hence $\tilde{t}_r \leq T^*_{r-1}$.

We then move to the case (II). Consider a hypothetical setting that there are only $r-1$ projects and the principal assigns the $r-1$-th-to-last project at stage $i+1$. Let $\Pi^{r-1}(j, i+1)$ be the part of E_P^m which are contributed by these $r-1$ projects. Denote $C_2 = \lambda_H\delta(1-\alpha_H)f_r(c_L)\frac{\delta^{t_{r+1}-1}\alpha_H^{t_{r+1}-1}}{1-\delta\alpha_H}$.

Notice that

$$\begin{aligned}
&\Pi^r(j,i) \\
= &\Pi^{r-1}(j, i+1) + \lambda_H\delta(1-\alpha_H)\sum_{t_{r+1}-1\leq k\leq i-2}\delta^k\alpha_H^k f_r(c_L)
\end{aligned}$$

[20] In particular,

$$\begin{aligned}
C_1 = &\lambda_H\delta(1-\alpha_H)f_r(c_L)\frac{\delta^{t_{r+1}-1}\alpha_H^{t_{r+1}-1}}{1-\delta\alpha_H} \\
&+ \lambda_H\delta(1-\alpha_H)\sum_{t_{r+1}-1\leq k\leq T^*_{r-1}-2}\delta^k\alpha_H^k\left[f_1(c_L)+\cdots+f_{r-1}(c_L)\right] \\
&+ \lambda_H\delta^{T^*_{r-1}-1}\alpha_H^{T^*_{r-1}-1}(v-c_H) \\
&+ \cdots \\
&+ \lambda_H\delta(1-\alpha_H)\sum_{T^*_2-1\leq k\leq T^*_1-2}\delta^k\alpha_H^k(v-c_H) \\
&+ \lambda_H\delta^{T^*_1-1}\alpha_H^{T^*_1-1}(v-c_H) \\
&- \lambda_L(c_H-c_L)\sum_{1\leq k\leq r-1}\delta^{T^*_k-1}(\alpha_H-\alpha_L)^{T^*_k-1},
\end{aligned}$$

which does not depend on i.

$$
\begin{aligned}
&+\lambda_H \delta^{i-1}\alpha_H^{i-1}(v-c_H) - \lambda_L(c_H-c_L)\delta^{i-1}\left(\alpha_H-\alpha_L\right)^{i-1} \\
=\,& \Pi^{r-1}(j,i+1) \\
&+\lambda_H\delta(1-\alpha_H)f_r(c_L)\frac{\delta^{t_{r+1}-1}\alpha_H^{t_{r+1}-1}}{1-\delta\alpha_H} - \lambda_H\delta(1-\alpha_H)f_r(c_L)\frac{\delta^{i-1}\alpha_H^{i-1}}{1-\delta\alpha_H} \\
&+\lambda_H \delta^{i-1}\alpha_H^{i-1}(v-c_H) - \lambda_L(c_H-c_L)\delta^{i-1}\left(\alpha_H-\alpha_L\right)^{i-1} \\
=\,& \Pi^{r-1}(j,i+1) + C_2 \\
&+\lambda_H\delta^{i-1}\alpha_H^{i-1}\left[(v-c_H) - \frac{\delta(1-\alpha_H)f_r(c_L)}{1-\delta\alpha_H} - \frac{\lambda_L}{\lambda_H}(c_H-c_L)\left(\frac{\alpha_H-\alpha_L}{\alpha_H}\right)^{i-1}\right].
\end{aligned}
$$

Due to our induction hypothesis and the fact that $i+1 > T^*_{r-1}$, $\Pi^{r-1}(j,i+1)$ must be less than the payoff contributed by the last $r-1$ projects via setting the deadline as the stage T^*_{r-1}. In addition, by the same argument in the case (I) above,

$$
\lambda_H\delta^{i-1}\alpha_H^{i-1}\big[v-c_H - \frac{\delta(1-\alpha_H)f_r(c_L)}{1-\delta\alpha_H} - \frac{\lambda_L}{\lambda_H}(c_H-c_L)\left(\frac{\alpha_H-\alpha_L}{\alpha_H}\right)^{i-1}\big]
$$

is increasing in terms of i up to a number (less than $\tilde{t}_r$, and hence also T^*_{r-1}), and then decreasing. As a result, the principal's expected payoff in case (II) is less than that in case (I). The principal should use the strategy in case (I), and hence set the deadline to assign the r-th-to-last project at stage $\tilde{t}_r$.[21] That is, $T^*_r = \tilde{t}_r$ is the smallest t such that

$$
\begin{aligned}
(v-c_H) \geq\,& \delta\left((v-c_H)\alpha_H + f_r(c_L)(1-\alpha_H)\right) \\
&+\frac{\lambda_L}{\lambda_H}(c_H-c_L)\left[1-\delta(\alpha_H-\alpha_L)\right]\left[\frac{\alpha_H-\alpha_L}{\alpha_H}\right]^{t-1}.
\end{aligned}
$$

This completes the induction argument.

Step 2. Next, we construct the payment rule and check that the corresponding mechanism satisfies all the incentive constraints and participation constraints.

Given c^{t-1} and s^t with $s_t = 1$,

$$
U(c^{t-1}, c_L; s^t) = E^m_A(c^{t-1}; s^t) \text{ and } U(c^{t-1}, c_H; s^t) = 0.
$$

Let

$$
\begin{aligned}
p_t(c^{t-1}, c_t; s^t) =\,& U(c^{t-1}, c_t; s^t) + q_t(c^{t-1}, c_t; s^t)c_t \\
&-\delta q_t(c^{t-1}, c_t; s^t)\mathbb{E}(U(c^{t-1}, c_t, c_{t+1}; \tilde{s}^{t+1})|c_t) \\
&-\delta(1-q_t(c^{t-1}, c_t; s^t))\mathbb{E}(U(c^{t-1}, c_t, c_{t+1}; \hat{s}^{t+1})|c_t).
\end{aligned}
$$

Then all the participation constraints are satisfied and the agent gets the payoff $U(c^{t-1}, c_t; s^t)$ at the cost c_t given c^{t-1} and s^t.

Following the argument in the proof of Lemma 4, the agent with low cost would

[21] Since the payoff in case (III) is the limit of the payoff in case (II), the principal will not choose the assignment rule in case (III).

not report high cost. We need to check that the agent with high cost would not report low cost. Given c^{t-1} and s^t, if the agent has cost the c_H but reports c_L, then her payoff is

$$
\begin{aligned}
& - q_t(c^{t-1}, c_L; s^t)c_H + p_t(c^{t-1}, c_L; s^t) \\
& + \delta q_t(c^{t-1}, c_L; s^t)\mathbb{E}\left[U(c^{t-1}, c_L, c_{t+1}; \tilde{s}^{t+1})|c_H\right] \\
& + \delta(1 - q_t(c^{t-1}, c_L; s^t))\mathbb{E}\left[U(c^{t-1}, c_L, c_{t+1}; \hat{s}^{t+1})|c_H\right] \\
= {} & U(c^{t-1}, c_L; s^t) - (c_H - c_L) \\
& - \delta\big[U(c^{t-1}, c_L, c_L; \tilde{s}^{t+1}) - U(c^{t-1}, c_L, c_H; \tilde{s}^{t+1})\big](\alpha_H - \alpha_L) \\
\leq {} & 0 \\
= {} & U(c^{t-1}, c_H; s^t).
\end{aligned}
$$

The first equality holds since $q_t(c^{t-1}, c_L; s^t) = 1$. The inequality holds since

$$E_A^m(c^{t-1}; s^t) - \delta(\alpha_H - \alpha_L)E_A^m(c^{t-1}, c_L; \tilde{s}^{t+1}) \leq c_H - c_L.$$

This completes the proof. $\square$