Discontinuous Stochastic Games

Wei He*

May 24, 2020

Abstract

We prove for a large class of stochastic games with discontinuous payoffs, a stationary Markov perfect equilibrium exists under the condition of "continuation payoff security." The condition is easy to verify and holds in many economic games. Roughly, a game belongs to this class if for any action/state profiles and continuation payoff, a player can identify another action at the current stage with the payoff not much worse than her current one, even if other players perturb actions slightly. As an illustrative application of the equilibrium existence result, we provide a stochastic dynamic oligopoly model of firm entry, exit, and price competitions.

Keywords: Stochastic game, discontinuous payoff, continuation payoff security, stationary Markov perfect equilibrium, dynamic oligopoly.

*Department of Economics, The Chinese University of Hong Kong, Hong Kong SAR, China. E-mail: hewei@cuhk.edu.hk

Contents

1 Introduction

Stochastic games have been proven very successful in modeling dynamic situations, and have found wide applications in economics and other disciplines.[1] Due to its simplicity and usefulness, the concept of stationary Markov perfect equilibrium is of central interest in the analysis of such games. Furthermore, it is often convenient to formulate strategic settings as games with general action spaces; for example, timing games, price and spatial competitions, auctions, bargaining, etc. The issue of payoff discontinuity arises naturally in these economic games. The purpose of this paper is to prove the existence of stationary Markov perfect equilibrium in discounted stochastic games with discontinuous payoffs.

The model of stochastic games was first introduced in Shapley (1953). A stochastic game is played by finitely many players in discrete time. At the beginning of each stage, some states are drawn randomly. After observing the common state and their own private states, players choose actions simultaneously. The game is then repeated at the next stage, where the distribution of the states is determined by the previous states and actions. Each player evaluates the outcome of the game via the discounted sum of stage payoffs. In general, a player's strategy is a complete plan that specifies the action chosen by the player for every history after which it is her turn to move. However, given the stationary structure of the model, it is natural to study stationary Markov strategies, which only depend on the current states rather than the entire past history and calendar time.[2] Stationary Markov strategies capture the time-independence feature of stochastic games. An equilibrium based on such strategies is called stationary Markov perfect equilibrium.

The existence of stationary Markov perfect equilibria in stochastic games remains an important problem in the literature. The attention has been concentrated on stochastic games either with finitely many actions, or with general action spaces and continuous payoffs. In the paper by Shapley (1953), the existence

[1]See, for example, the survey chapters by Amir (2003), Vives (2005), and Jaśkiewicz and Nowak (2018a,b).

[2]As noted in Maskin and Tirole (2001), a virtue of the concept of Markov perfect strategies is that they are conceptually and computationally simple.

result was established for finite-action, finite-state, zero-sum stochastic games. Fink (1964) and Takahashi (1964) proved the equilibrium existence in stochastic games with compact action spaces but finitely many states. For the games with general action and state spaces, various conditions have been imposed on the state structure to obtain the equilibrium existence; see, for example, Nowak and Raghavan (1992), Duffie *et al.* (1994), Duggan (2012), and He and Sun (2017).[3] All these papers working with general action spaces assume continuous payoffs. However, as discussed in the beginning, many games introduced in economics exhibit discontinuities in payoff. Yet, there is no existence result of stationary Markov perfect equilibria for stochastic games with discontinuous payoffs.

The present paper provides a general existence result for stationary Markov perfect equilibria in a large class of discontinuous stochastic games. Towards this end, we introduce the condition of continuation payoff security. By the standard dynamic programming method, one can reformulate a player's problem in a stochastic game via the Bellman equation, in which the player's utility is a combination of her current stage's payoff and the discounted continuation payoff. Intuitively, the condition of continuation payoff security means that given the continuation payoff, a player has an action that gives herself a payoff not much worse than the current one, even if other players' actions are perturbed slightly. The condition is easy to verify in practical examples, which is important for theoretical applications. Based on the continuation payoff security condition, we show in Theorem 1 that a stationary Markov perfect equilibrium exists.

As stochastic games have become the workhorse framework for analyzing the industry dynamics of oligopoly markets,[4] we provide a dynamic model of firm entry, exit, and price competition in an industry to illustrate the application of the equilibrium existence theorem. In the application, the price competitions happen in multiple markets and firms naturally face the issue of payoff discontinuity. This

[3] Nowak and Raghavan (1992) and Duffie *et al.* (1994) introduced a randomization device and obtained the existence of stationary correlated equilibria. Duggan (2012) showed the equilibrium existence for a class of stochastic games with a product structure on the states. He and Sun (2017) imposed the condition of "coarser transition kernel" and proved the equilibrium existence result. In general, a stationary Markov perfect equilibrium may fail to exist for stochastic games with a continuum of states, even when the payoffs are continuous in actions; see Levy (2013) and Levy and McLennan (2015).

[4] For some classical references, see Maskin and Tirole (1988a,b) and Ericson and Pakes (1995).

dynamic oligopoly model cannot be covered by the extant literature. It is shown that such a model is a continuation payoff secure stochastic game, and possesses a stationary Markov perfect equilibrium.

There is a large literature on discontinuous games in the last two decades.[5] The mostly related works in this literature to the current paper are Reny (1999) and Carbonell-Nicolau and McLean (2018). In particular, the condition of continuation payoff security extends the payoff security condition in Reny (1999) for normal form games, and the uniform payoff security condition in Carbonell-Nicolau and McLean (2018) for Bayesian games. Surprisingly, all the previous works, including these two papers, focus on the static environment, while little is known about the equilibrium existence for games with payoff discontinuities in the dynamic environment. On the other hand, we work with discontinuous stochastic games in the current paper.

A standard proof method for the equilibrium existence result in static discontinuous games can be described as follows. Based on the payoff security-type condition, one can approximate the discontinuous payoff functions by a sequence of continuous payoff functions from below, which yields an appropriate sequence of strategy profiles. Some weaker version of the upper semicontinuity condition on the payoffs then guarantees that the limit of the sequence of those strategy profiles is indeed an equilibrium of the original discontinuous game. This idea has been employed by Reny (1999) and many other papers.[6]

The proof techniques as described above are not readily applicable in our setting. New difficulties arise for two reasons. First, given the dynamic nature of the model, it is very hard, if not impossible, to handle the discounted sum of players' stage payoffs directly. Instead, we work with the utility as a combination of stage payoff and discounted continuation payoffs, which is based on the

[5]Since the important contribution by Reny (1999), discontinuous games have been studied in various settings; see, for example, Monteiro and Page (2007) and Prokopovych and Yannelis (2014) for the mixed-extension of normal form games, Carbonell-Nicolau and McLean (2018), He and Yannelis (2016), and Prokopovych and Yannelis (2019) for Bayesian games, and Reny (2016) and Carbonell-Nicolau and McLean (2019) for ordinal games. The literature is too vast to be discussed in the context of this paper, we refer the readers to the recent survey by Reny (2020).

[6]See Prokopovych (2011) for further elaborations of the idea. Carbonell-Nicolau and McLean (2018) and He and Yannelis (2016) adopt this argument to prove the existence of Bayesian equilibria in games with incomplete information. Besides the payoff security-type conditions, they also impose the condition of aggregate upper semicontinuity from Dasgupta and Maskin (1986), which requires the sum of players' payoffs be upper semicontinuous.

standard dynamic programming argument. Note that the continuation payoff is endogenously generated by the strategy profiles via the Bellman equation, which makes it hard to analyze given the payoff discontinuity issue. To circumvent this difficulty, we observe that the continuation payoff is lower semicontinuous and can be approximated by a sequence of continuous mappings. Importantly, the contraction property on the continuation payoffs in the Bellman equation is kept in the approximations. The second difficulty is that one cannot immediately claim the limit of some appropriate sequence of stationary strategy profiles to be an equilibrium, even under the aggregate upper semicontinuity condition. This is due to the fact that each player's utility depends on the lower semicontinuous continuation payoffs, and the conflicting effects of lower and upper semicontinuities make it unclear whether the sum of utilities is upper semicontinuous or not. The key observation is that though we are not sure whether the aggregate utilities is upper semicontinuous in general, it is indeed upper semicontinuous at the limit of a carefully chosen sequence of stationary strategy profiles. Combining these two points, we are able to analyze dynamic games with discontinuous payoffs and obtain the equilibrium existence result via the convenient recursive framework.

The rest of the paper is organized as follows. Section 2 describes the model and the concept of stationary Markov perfect equilibrium. Section 3 introduces the continuation payoff security condition and proves the equilibrium existence result. An illustrative application of dynamic oligopoly model is presented in Section 4. In Section 5, we sketch the key ideas of the proof of Theorem 1. The details of the proofs are left in Appendix.

2 Model

2.1 Stochastic games

An n-person incomplete-information stochastic game can be described by (1) a common state space, (2) players' private state spaces, (3) a state-dependent feasible action correspondence for each player, (4) stage payoffs that depends on the state profile and action profile, (5) a discount factor for each player, and (6) a transition probability that depends on the state and action profiles.

Formally, an incomplete-information discounted stochastic game is formulated as follows.

- The set of players is $I = \{1, \cdots, n\}$.
- The common state of nature is drawn from the set $\Omega = \{\omega^1, \dots, \omega^K\}$.
- For each $i \in I$, player i draws some private state s_i from S_i, where S_i is a Polish space endowed with the σ-algebra $\mathcal{S}_i$. Denote $S = \prod_{i \in I} S_i$ and $\mathcal{S} = \otimes_{i \in I} \mathcal{S}_i$.
- The action space of player i is X_i, a nonempty compact metric space endowed with its Borel σ-algebra $\mathcal{B}(X_i)$. Denote $X = \prod_{i \in I} X_i$ and $\mathcal{B}(X) = \otimes_{i \in I} \mathcal{B}(X_i)$.
- Given a common state ω, the set of feasible actions of player i is $A_i(\omega)$, which is a nonempty and compact subset of X_i. Let $A(\omega) = \prod_{i \in I} A_i(\omega)$ for each $\omega \in \Omega$.
- Player i's stage payoff u_i is a bounded measurable mapping from $\Omega \times S \times X$ to $\mathbb{R}$. We assume that for some $C > 0$, $|u_i(\omega, \mathbf{s}, \mathbf{x})| \le C$ for any $(\omega, \mathbf{s}, \mathbf{x}) \in \Omega \times S \times X$, where $\mathbf{s}$ and $\mathbf{x}$ are the state profile and action profile, respectively.
- The discount factor of player i is $\beta_i \in [0, 1)$.
- The transitions of common states and private states are described in the followings.
  1. The law of motion of the common states is given by $Q \colon \Omega \times X \to \mathcal{M}(\Omega)$.[7] That is, if ω is the common state at stage t and $\mathbf{x} \in X$ is the action profile chosen by the n players at this stage, then $Q(\omega'|\omega, \mathbf{x})$ (abbreviated as $Q_{(\omega,\mathbf{x})}(\omega')$) is the probability that the common state is ω' at stage $t + 1$.
  2. Conditional on the common state ω' at stage $t + 1$, the n players draw private states $(s'_1, \dots, s'_n)$ at stage $t + 1$ independently. In particular, a private state s'_i of player i is taken from a measurable set $E \subseteq S_i$ based on the probability $P_i(E|\omega')$. Let λ_i be a probability measure on $(S_i, \mathcal{S}_i)$ such that $P_i(\cdot|\omega')$ is absolutely continuous with respect to λ_i for each

[7] For a Borel set D, let $\mathcal{M}(D)$ be the set of all Borel probability measures on D.

$\omega' \in \Omega$, and $\mathcal{S}_i$ the completion of the Borel σ-algebra on S_i under λ_i.[8]

Remark 1. *We assume that there are finitely many common states while allowing for general private state spaces. This is natural for applications. In many economic settings, the common state represents the status of the environment or the players (e.g., the market being in boom or recession, firms being active or not, or the occupational choices of agents), which is typically a discrete variable. To the contrary, the private state of a player may describe the player's private value, belief or other information, which is taken from a general space. For example, in a dynamic oligopoly market with finitely many firms as considered in Section 4, a common state describes the market position (incumbent or potential entrant) of firms. On the one hand, there are finitely many such common states, as the number of firms is finite. On the other hand, the private state of a firm represents its private information, such as scrap values or production shocks, while is naturally modelled as a number taken from the real line.*

Up to stage $t \geq 1$, a public history of a stochastic game is

$$h_t = (\omega_1, \mathbf{x}_1, \ldots, \omega_{t-1}, \mathbf{x}_{t-1}, \omega_t),$$

where the action profile $\mathbf{x}_j \in A(\omega_j)$ and $\omega_j \in \Omega$ for stage $1 \leq j \leq t-1$. Let H_t be the space of all public histories h_t, which represents the information commonly known to all players before they take actions at stage t. Besides h_t, player i observes a private state s_{ij} at each stage j, which is not observable to the other players. The private history of player i at stage t is

$$h_{it} = (\omega_1, s_{i1}, \mathbf{x}_1, \ldots, \omega_{t-1}, s_{i(t-1)}, \mathbf{x}_{t-1}, \omega_t, s_{it}).$$

Let H_{it} be the space of all private histories of player i at stage t.

After observing the private history h_{it}, player i chooses an action from the set of feasible actions $A_i(\omega_t)$. Formally, at stage $t \geq 1$, a strategy f_i of player i specifies

[8] It is without loss of generality to assume that there exists a reference probability measure λ_i on $(S_i, \mathcal{S}_i)$. For concreteness, let $\lambda_i(E) = \frac{1}{K}\sum_{1\leq k\leq K} P_i(E|\omega^k)$ for any measurable set $E \subseteq S_i$. Then $P_i(\cdot|\omega^k)$ is absolutely continuous with respect to λ_i for $1 \leq k \leq K$, and hence possesses a density.

a measurable mapping from H_{it} to $\mathcal{M}(X_i)$, placing probability 1 on the feasible action set $A_i(\omega_t)$ for $\omega_t \in \Omega$.

Given a strategy profile $f = (f_1, \ldots, f_n)$, player i's expected payoff is $\mathbb{E}^{f}_{(\omega_1, s_{i1})} \left(\sum_{t=1}^{\infty} \beta_i^{t-1} u_i(\omega_t, \mathbf{s}_t, \mathbf{x}_t) \right)$, when the common state is ω_1 and her private state is s_{i1} in the very beginning.

2.2 Stationary Markov perfect equilibrium

Hereafter, we shall focus on a special class of strategies, namely the stationary Markov strategies. The consideration of stationary Markov strategies in stochastic games is common. A strategy of player i is called a stationary Markov strategy if at each stage $t \geq 1$, her (mixed) action plan only depends on the current common state ω_t and private state s_{it}, but not on the previous history and calendar time t. As highlighted in Maskin and Tirole (2001), Markov strategies have the advantage that players only need to depend on payoff-relevant data when choosing actions. In addition, stationary Markov strategies are practically useful as they are easy to analyze.

A stationary Markov strategy of player i is a measurable mapping $f_i \colon \Omega \times S_i \to \mathcal{M}(X_i)$ such that $f_i(\omega, s_i)$ places probability 1 on the feasible action set $A_i(\omega)$ for each $(\omega, s_i) \in \Omega \times S_i$. For $1 \leq k \leq K$, let $\mathcal{L}_i^k$ be the set of all measurable mappings from S_i to $\mathcal{M}(A_i(\omega^k))$, which is endowed with the following topology of weak convergence: for any sequence $\{g_j\}_{j \geq 0} \subseteq \mathcal{L}_i^k$, $g_j \to g_0$ if

$$\int_{S_i} \int_{A_i(\omega^k)} \gamma(s_i, x_i) g_j(\mathrm{d}x_i | s_i) P_i^k(\mathrm{d}s_i) \to \int_{S_i} \int_{A_i(\omega^k)} \gamma(s_i, x_i) g_0(\mathrm{d}x_i | s_i) P_i^k(\mathrm{d}s_i)$$

for any bounded function γ that is measurable in s_i and continuous in x_i. Then $\mathcal{L}_i = \prod_{1 \leq k \leq K} \mathcal{L}_i^k$ is the set of all stationary Markov strategies for player i. As usual, $\mathcal{L} = \prod_{i \in I} \mathcal{L}_i$.

A stationary Markov strategy profile $f = (f_1, \ldots, f_n)$ is called a **stationary Markov perfect equilibrium** if

$$\mathbb{E}^{f}_{(\omega_1, s_{i1})} \left(\sum_{t=1}^{\infty} \beta_i^{t-1} u_i(\omega_t, \mathbf{s}_t, \mathbf{x}_t) \right) \geq \mathbb{E}^{(g_i, f_{-i})}_{(\omega_1, s_{i1})} \left(\sum_{t=1}^{\infty} \beta_i^{t-1} u_i(\omega_t, \mathbf{s}_t, \mathbf{x}_t) \right)$$

for any player $i \in I$, common state $\omega_1 \in \Omega$, private state $s_{i1} \in S_i$, and i's (not necessarily Markov) strategy g_i. The notation $-i$ means all the players except i.

In the following, we reformulate the notion of stationary Markov perfect equilibrium via the standard Bellman equation, which is much easier to work with. Formally, given a stationary Markov strategy profile f, player i's continuation payoff $v_i(\cdot, \cdot, f)$ (abbreviated as $v_i^f(\cdot, \cdot)$) gives an essentially bounded measurable mapping from $\Omega \times S_i$ to $\mathbb{R}$, which is uniquely determined by the following recursion[9]

$$\begin{aligned} & v_i^f(\omega, s_i) \\ = & \int_{S_{-i}} \int_X \Big[u_i(\omega, s_i, s_{-i}, x_i, x_{-i}) \\ & + \beta_i \sum_\Omega \int_{S_i} v_i^f(\omega', s_i') P_i(\mathrm{d}s_i'|\omega') Q(\omega'|\omega, \mathbf{x}) \Big] \otimes_{j \in I} f_j(\mathrm{d}x_j|\omega, s_j) P_{-i}(\mathrm{d}s_{-i}|\omega). \end{aligned} \tag{1}$$

Given any continuation payoffs $v = (v_1, \dots, v_n)$ and common state ω, one can construct an auxiliary Bayesian game $\Gamma(v, \omega)$. The action space for player i is $A_i(\omega)$. The private state space of player i is $(S_i, \mathcal{S}_i, P_i(\cdot|\omega))$. Given the state profile $\mathbf{s}$ and the action profile $\mathbf{x} \in A(\omega)$, the payoff of player i is

$$U_i(\omega, \mathbf{s}, \mathbf{x}, v_i) = u_i(\omega, \mathbf{s}, \mathbf{x}) + \beta_i \sum_\Omega \int_{S_i} v_i(\omega', s_i') P_i(\mathrm{d}s_i'|\omega') Q(\omega'|\omega, \mathbf{x}).$$ [10]

A strategy profile f is a **stationary Markov perfect equilibrium** if $f(\omega, \cdot)$ is a Bayesian equilibrium in the auxiliary Bayesian game $\Gamma(v^f, \omega)$ for every ω. It implies that the continuation value v_i^f coincides with the equilibrium payoff in the auxiliary Bayesian game, and hence solves the recursive maximization problem:

$$\begin{aligned} & v_i^f(\omega, s_i) \\ = & \sup_{x_i \in A_i(\omega)} \int_{S_{-i}} \int_{X_{-i}} U_i^{v_i^f}(\omega, s_i, s_{-i}, x_i, x_{-i}) \otimes_{j \neq i} f_j(\mathrm{d}x_j|\omega, s_j) P_{-i}(\mathrm{d}s_{-i}|\omega) \\ = & \sup_{x_i \in A_i(\omega)} \int_{S_{-i}} \int_{X_{-i}} \Big[u_i(\omega, s_i, s_{-i}, x_i, x_{-i}) \end{aligned}$$

[9]It is easy to show the existence and uniqueness of the continuation value via a standard contraction mapping argument; see, for example, Blackwell (1965) and Jaśkiewicz and Nowak (2018a,b) for standard results in dynamic programming and stochastic games.

[10]For simplicity, $U_i(\omega, \mathbf{s}, \mathbf{x}, v_i)$ is also abbreviated as $U_i^{v_i}(\omega, \mathbf{s}, \mathbf{x})$.

$$+ \beta_i \sum_{\Omega} \int_{S_i} v_i^f(\omega', s_i') P_i(\mathrm{d}s_i'|\omega') Q(\omega'|\omega, x_i, x_{-i}) \Bigg] \otimes_{j \neq i} f_j(\mathrm{d}x_j|\omega, s_j) P_{-i}(\mathrm{d}s_{-i}|\omega). \tag{2}$$

3 Main results

Stochastic games are the natural framework for many applications, including dynamic oligopolies, evolutionary biology and computer networks. Among these applications, the issue of payoff discontinuity arises naturally. It would be desirable if some appropriate conditions on payoff discontinuity can be incorporated into the framework of stochastic games, while the existence of stationary Markov perfect equilibria is guaranteed. Towards this aim, we shall introduce the notion of "continuation payoff security," which captures the payoff discontinuity in a wide class of applications. Based on this condition, the existence of stationary Markov perfect equilibria is shown in Theorem 1.

Before moving to the statement of the condition and the result, we first describe the set of possible continuation payoffs. For $i \in I$, let $L_{\infty i}$ be the L_∞ space of all $\mathcal{S}_i$-measurable mappings from S_i to the Euclidean space $\mathbb{R}^K$ under the probability measure λ_i; that is,

$$L_{\infty i} = \{v_i \colon v_i \text{ is } \mathcal{S}_i\text{-measurable and essentially bounded under } \lambda_i\}.$$

Denote $V_i = \{v_i \in L_{\infty i} \colon \|v_i\|_\infty \leq \frac{C}{1-\beta_i}\}$, where C is the bound of the stage payoff and $\|\cdot\|_\infty$ is the standard essential sup norm. Then V_i contains the set of all possible continuation payoffs for player i. Denote V_i^k as the k-th dimension of V_i; that is, V_i^k contains the set of possible continuation payoffs for player i when the common state is ω^k.

The notion of "continuation payoff security" is introduced in the following.

Definition 1. *A stochastic game is called continuation payoff secure if for any player i, $\epsilon > 0$, $f_i \in \mathcal{L}_i$, there exists some $g_i \in \mathcal{L}_i$ such that for any $(\omega, s_i, s_{-i}, x_{-i}, v_i) \in \Omega \times S_i \times S_{-i} \times X_{-i} \times V_i$ with $x_{-i} \in A_{-i}(\omega)$, there exists* *some neighborhood $O_{x_{-i}}$ of x_{-i} with*

$$U_i(\omega, s_i, s_{-i}, g_i(\omega, s_i), y_{-i}, v_i) > U_i(\omega, s_i, s_{-i}, f_i(\omega, s_i), x_{-i}, v_i) - \epsilon$$

for any $y_{-i} \in O_{x_{-i}}$.

Remark 2. *The condition of continuation payoff security roughly means that given any action/state profiles and continuation payoffs, a player can identify another action at this stage that secures herself a payoff not much worse than the given payoff, even if other players' actions are perturbed slightly. Given the payoff functions and state transitions, it is straightforward to verify the condition of continuation payoff security. It is clear that this condition holds if the payoff and state transition are continuous. In general, to secure a payoff, this condition requires each player take the continuation payoff as given and only identify an action at one stage.*[11] *This is considerably simpler than identifying a strategy that specifies a complete plan for all future subgames. In applications, this condition is often easy to be satisfied. For example, in auctions or price competitions, the payoff discontinuity at some stage is typically due to a tie among players' action profile at that stage, and a player can secure the payoff by slightly reducing her bid/price.*

The continuation payoff security condition generalizes the payoff security condition in Reny (1999), and the uniform payoff security condition in Carbonell-Nicolau and McLean (2018), which shall be discussed in details in Section 5. These papers focus on the static setting, while we study the dynamic setting.

Next, we define the standard condition of "aggregate upper semicontinuity." This condition has been widely studied in the literature; see, for example, Dasgupta and Maskin (1986) and Reny (1999) for normal-form games, and Carbonell-Nicolau and McLean (2018) and Prokopovych and Yannelis (2019) for Bayesian games.

Definition 2. *A stochastic game is called aggregate upper semicontinuous if for any $(\omega, \mathbf{s}, v)$, $\sum_{i \in I} U_i(\omega, \mathbf{s}, \cdot, v_i)$ is upper semicontinuous in $\mathbf{x}$.*

[11]This is natural as we consider dynamic games. For example, by the one-shot deviation principle, to verify whether a given strategy profile is a subgame-perfect equilibrium or not, it suffices to check that no player can benefit by deviating from the strategy only for one stage while taking the continuation payoff as given; that is, reverting to the strategy thereafter.

The condition of aggregate upper semicontinuity requires the summation of the utility functions of all players be upper semicontinuous. It is weaker than the upper semicontinuity condition of the utility functions. The payoff discontinuity in applications, such as auction or price competitions, is often due to a tie among some players. Though the payoff u_i of player i could be discontinuous in such a case, the summation $\sum_{i \in I} u_i$ in these economic examples is often continuous in the action profile $\mathbf{x}$.

We now state the main result of the paper.

Theorem 1. *If a stochastic game is continuation payoff secure and aggregate upper semicontinuous, then it possesses a stationary Markov perfect equilibrium.*

The proof of Theorem 1 is sketched in Section 5. The complete proof is left in Section 6.1.

4 An application

As an illustrative example, we provide in this section a general model of dynamic oligopoly with entry and exit. Firms play in discrete time with an infinite horizon. A firm can be either an incumbent firm or a potential entrant. An incumbent firm has to decide whether to remain in the industry and, if so, the price decisions. An inactive firm decides the prices of the products if it chooses to enter the industry. A firm's profit depends on the decisions of all firms and also the market shocks. The model is comparable to those in Maskin and Tirole (1988a,b), Ericson and Pakes (1995), and Doraszelski and Satterthwaite (2010). We shall verify that such a stochastic dynamic oligopoly model is a continuation payoff secure stochastic game. Proposition 1 below claims the existence of a stationary Markov perfect equilibrium.

Formally, there are n firms in the market with $I = \{1, \ldots, n\}$. Let $\Omega_i = \{0, 1, \ldots, \kappa\}$ indicate the status of firm i. A state ω_i in $\{1, \ldots, \kappa\}$ represents the status of firm i when it is active; that is, firms can differ from each other in more than one aspect. State 0 identifies firm i as being inactive. The vector $\omega = (\omega_1, \ldots, \omega_n)$ describes the market state, and $\Omega = \{0, 1, \ldots, \kappa\}^n$ is the common state space with $K = (\kappa+1)^n$. Given the market state ω, firms draw private states $(s_1, \ldots, s_n) \in \prod_{i \in I}[\underline{s}_i, \overline{s}_i]$ independently based on $\otimes_{i \in I} P_i(\cdot|\omega)$, where $\overline{s}_i \geq \underline{s}_i \geq 0$ for each $i \in I$. If firm i is an incumbent firm, then s_i is the scrap value it receives upon exit; if firm i is a potential entrant, then s_i is the cost it needs to pay when entering the industry.

In many industries, firms often compete with each other in all low-end, mid-tier, and high-end markets, even though they may have different emphasis and advantages in different segments of the market. To capture such feature, we assume that firms set prices for L heterogeneous products. For $1 \leq l \leq L$, the price of the l-th product is chosen from the interval $[\underline{a}_l, \overline{a}_l]$ with $\overline{a}_l > \underline{a}_l \geq 0$. At each stage, firms observe the entry, exit, and price decisions from the previous stages, as well as the market state ω at the beginning of the current stage. In addition, an incumbent firm learns the scrap value, and decides the exit and price decisions (if it remains in the market), while a potential entrant draws the cost, and decides the entry and price decisions (if it decides to enter the market). Specifically, the action space of firm i is $X_i = \prod_{1 \leq l \leq L}[\underline{a}_l, \overline{a}_l] \cup \{-1\}$ for each $i \in I$. A vector in $\prod_{1 \leq l \leq L}[\underline{a}_l, \overline{a}_l]$ represents the prices that the firm chooses for the L products, while the action -1 for an incumbent firm (resp. a potential entrant) means that the firm exits (resp. chooses not to enter) the market. For each $i \in I$, firm i discounts future payoffs using a discount factor β_i.

Given the market state ω and the actions $(x_1, \ldots, x_n)$ from all firms at the current stage, the market state ω' at the next stage is drawn based on the law of motion $Q(\omega'|\omega, x_1, \ldots, x_n)$, which is a continuous mapping from $\Omega \times \prod_{i \in I} X_i \to \mathcal{M}(\Omega)$. In particular, if an incumbent firm i chooses to exit the market or a potential entrant i decides not to enter the market (*i.e.*, $x_i = -1$), then its market state ω_i' at the next stage is 0; otherwise, ω_i' is drawn randomly from $\{1, \ldots, \kappa\}$.[12]

For product $1 \leq l \leq L$, the market demand is given by a continuous mapping $D_l \colon \Omega \times \prod_{i \in I} X_i \to \mathbb{R}_+$. Note that the demand for product l not only depends on

[12] A firm could choose its market position over time through price decisions. Intuitively, a firm with lower prices is able to possess a greater market share at the current stage. At the future stages, this firm is more likely to be in an advantageous position. For example, a big firm is known by more consumers, is accessible in more submarkets, and is often more likely to be chosen by consumers when the price of its product is the same as the price of the product from some small firm.

all the firms' price decisions of this product, but also on the prices of the other products and the market state. For example, when the market price of a mid-tier smartphone is not a lot higher than that of a low-end smartphone, there could be a significant proportion of consumers who choose the former rather than the latter. Given the market state ω, the cost of production for the l-th product is $c_l(\omega) \in (0, \overline{a}_l)$.[13]

1. Let firm i be an incumbent firm.
   - If $x_i = -1$, then firm i exits the market and receives the scrap value s_i.
   - If $x_i = (a_{i1}, \dots, a_{iL}) \neq -1$, then in each submarket for good l, the active firms (*i.e.*, the incumbent firms remaining in the market as well as the potential entrants entering the market) with the lowest price share the market. Firm i's payoff is the summation of the profits from all the products, and is given by

$$u_i(\omega, \mathbf{s}, \mathbf{x}) =$$

$$\sum_{1 \le l \le L, a_{il} = \min_{\{j \in I:\, x_j \neq -1\}} a_{jl}} \frac{\xi_{il}(\omega, x_1, \dots, x_n) D_l(\omega, \mathbf{x})}{\sum_{j' \in I: x_{j'} \neq -1, a_{j'l} = a_{il}} \xi_{j'l}(\omega, x_1, \dots, x_n)} (a_{il} - c_l(\omega)),$$

     where $\xi_l = (\xi_{1l}, \dots, \xi_{nl}) \colon \Omega \times \prod_{i \in I} X_i \to (0, 1]^n$ is a continuous mapping that assesses the relative importance of each firm's position when sharing the market.[14] In particular, if $\xi_{il} \equiv 1$ for any $i \in I$ and $1 \le l \le L$, then each submarket is shared equally. However, this is not necessary.

2. Let firm i be a potential entrant.
   - If $x_i = -1$, then firm i chooses not to enter the market and gets the payoff 0.
   - If $x_i = (a_{i1}, \dots, a_{iL}) \neq -1$, then firm i's payoff mapping is the same as that of an incumbent firm choosing to remain in the market, except that the entrant needs to pay the additional entry cost s_i.

[13] The dependence of the production cost on the market state allows for the interpretation that firms with different market states may have different bargaining power in the supply chain, and hence may face different production costs.

[14] For example, among all firms with the lowest price, firms with greater market power may be able to attract more consumers and enjoy larger market shares.

In the proposition below, we show that this stochastic dynamic oligopoly model can be viewed as a continuation payoff secure stochastic game satisfying the aggregate upper semicontinuity condition. Then a stationary Markov perfect equilibrium follows from Theorem 1.

Proposition 1. *The dynamic oligopoly model above possesses a stationary Markov perfect equilibrium.*

Remark 3. *The model above can be easily extended to more general settings. For example, in the important work of Ericson and Pakes (1995), an incumbent firm needs to make exit and investment decisions, which influence the transitions of the market state at future stages. Importantly, the firm's stage profit is a reduced form only depending on the market state (but not on its investment decision), which reflects the equilibrium in the industry spot market. In their paper, the stage payoff and the law of motion are both continuous in terms of the investment decisions. In this paper, we explicitly consider the price competition among firms and need to handle the payoff discontinuity issue. It is straightforward to extend our model to the setting allowing firms to make the investment decisions.*

5 Discussions of the proof of Theorem 1

5.1 Payoff security & uniform payoff security

The condition of continuation payoff security generalizes the condition of payoff security in Reny (1999) for normal form games, and the condition of uniform payoff security in Carbonell-Nicolau and McLean (2018) for Bayesian games.[15]

Consider a normal form game $(X_i, u_i)_{i \in I}$, where X_i and u_i are the action space and payoff function of player i, respectively.

Definition 3 (Payoff security)**.** *We say player i can secure a payoff $\alpha \in \mathbb{R}$ at $(x_i, x_{-i}) \in X_i \times X_{-i}$ if there is some $\overline{x_i} \in X_i$ such that $u_i(\overline{x_i}, y_{-i}) \geq \alpha$ for any y_{-i} in some open neighborhood of x_{-i}.*

[15] Monteiro and Page (2007) introduce the uniform payoff security condition in normal form games to prove that the mixed extension of a normal form game is payoff secure. This condition is further developed in Prokopovych and Yannelis (2014) for normal form games, and is generalized by Carbonell-Nicolau and McLean (2018) to Bayesian games.

The payoff u_i of player i is "payoff secure" if for any $(x_i, x_{-i}) \in X_i \times X_{-i}$ and $\epsilon > 0$, player i can secure a payoff $u_i(x_i, x_{-i}) - \epsilon$ at (x_i, x_{-i}). The game is called payoff secure if u_i is payoff secure for each player $i \in I$.

Next, consider a Bayesian game $\left((X_i, S_i, u_i)_{i\in I}, \lambda\right)$, where X_i, S_i and u_i are the action space, private state space, and payoff function of player i, and λ is the common prior on $S = \prod_{i\in I} S_i$.

Definition 4 (Uniform payoff security). *A Bayesian game is called "uniformly payoff secure" if for each $i \in I$, $\epsilon > 0$, and strategy f_i, there exists another strategy g_i such that for all (s, x_{-i}),*

$$u_i(s, g_i(s_i), y_{-i}) > u_i(s, f_i(s_i), x_{-i}) - \epsilon$$

for any y_{-i} in some open neighborhood of x_{-i}.

5.2 A roadmap for proving Theorem 1

To provide a general graph, we first describe the main objective of the proof.

For $i \in I$, $1 \le k \le K$, $f = (f_i, f_{-i}) \in \mathcal{L}$, and $v \in V$, let

$$R_i^k(f_i, f_{-i}, v_i) = \int_S \int_X U_i(\omega^k, \mathbf{s}, \mathbf{x}, v_i) \otimes_{m\in I} f_m(\mathrm{d}x_m|\omega^k, s_m) P(\mathrm{d}\mathbf{s}|\omega^k).$$

Recall the auxiliary Bayesian game $\Gamma(v, \omega^k)$ constructed in Section 2.2. It is easy to see that $R_i^k(f_i, f_{-i}, v_i)$ is the ex ante payoff of player i in this Bayesian game. Rather than working with the set V_i, we shall introduce an auxiliary set of continuation values $\tilde{V}_i \subseteq \mathbb{R}^K$. The k-th dimension of $\tilde{V}_i$, $\tilde{V}_i^k = [-\frac{C}{1-\beta_i}, \frac{C}{1-\beta_i}]$, representing the set of possible expected continuation values of player i when the common state is ω^k at the current stage; that is, given a continuation payoff $v_i \in V_i$,

$$\tilde{v}_i^k = \int_{S_i} v_i(\omega^k, s_i) P_i(\mathrm{d}s_i|\omega^k) \in \tilde{V}_i^k.$$

By slightly abusing the notation, $\tilde{v}_i \in \tilde{V}_i$ is also viewed as a continuation payoff in V_i that does not depend on s_i. We shall construct a mapping φ_i from $\mathcal{L}_{-i}$ to $\tilde{V}_i$,[16]

[16]The details of the construction of φ_i will be given later.

and define

$$W_i^k(f_i, f_{-i}) = R_i^k(f_i, f_{-i}, \varphi_i(f_{-i})).$$

To prove Theorem 1, the key is to show that there exists a stationary Markov strategy profile $f^* = (f_1^*, \ldots, f_n^*) \in \mathcal{L}$ such that for each $i \in I$ and $1 \le k \le K$,

1. $$\varphi_i^k(f_{-i}^*) = W_i^k(f_i^*, f_{-i}^*) \ge W_i^k(g_i, f_{-i}^*)$$

   for every $g_i \in \mathcal{L}_i$;

2. and
   $$\varphi_i^k(f_{-i}^*) = \int_{S_i} v_i^{f^*}(\omega^k, s_i) P_i(\mathrm{d}s_i | \omega^k),$$

   where $v_i^{f^*}$ is determined by the Bellman equation (1) in Section 2.2.

Roughly, $\varphi_i^k(f_{-i}^*)$ is player i's maximally possible expected payoff at state ω^k, if other players adopt the stationary strategies f_{-i}^* and player i adopts the optimal strategy in the future. Conditions (1) and (2) above imply that player i can achieve this optimal expected payoff by playing the stationary strategy f_i^*. It is clear that f^* is a stationary Markov perfect equilibrium.

Remark 4. *Before we proceed to the details, to better understand the proof of Theorem 1, we consider a straightforward approach one might try.*

Given the continuation payoff $v = (v_1, \ldots, v_n) \in V$ and common state ω^k, consider the auxiliary Bayesian game $\Gamma(v, \omega^k)$. First suppose that a Bayesian equilibrium exists in this game.[17] *Write $f = (f^1, \ldots, f^k)$, where f^k is a Bayesian equilibrium in the game $\Gamma(v, \omega^k)$. One can work with the correspondence that maps v to the set of all v^f, where v^f is the continuation payoff determined by some Bayesian equilibrium f via the Bellman equation (1) in Section 2.2. This is a correspondence from V to itself. If it has a fixed point v^f, then it is straightforward to see that the associated f is a stationary Markov perfect equilibrium.*[18]

[17] As shown in Milgrom and Weber (1985), a mixed-strategy Bayesian equilibrium exists when the payoffs are continuous; see also Fu (2008). To the contrary, a pure-strategy Bayesian equilibrium may fail to exist; see Khan, Rath and Sun (1999). Given the discontinuity of the payoffs and state transitions in the original stochastic game, the payoffs in the Bayesian game $\Gamma(v, \omega^k)$ may not be continuous.

[18] This is the standard approach to prove the equilibrium existence for stochastic games with complete information and continuous payoffs; see, for example, Nowak and Raghavan (1992).

Our approach is different. In the beginning of this subsection, even though we have not given the details of the proof yet, from the construction of W_i^k it is clear that the continuation value v_i of player i only depends on the strategies f_{-i} of the other players. This is different from the way to generate v_i^f by the strategy profile f via the Bellman equation (1). The reason we do not adopt the approach as described in the above paragraph is due to the following issues.

The first obvious difficulty is that the existence of Bayesian equilibria in the Bayesian game $\Gamma(v, \omega^k)$ is not immediate, as we consider stochastic games with discontinuous payoffs. This difficulty can be bypassed via the conditions of continuation payoff security and aggregate upper semicontinuity. The condition of continuation payoff security for a stochastic game implies the uniform payoff security for the Bayesian game $\Gamma(v, \omega^k)$. It is shown in Carbonell-Nicolau and McLean (2018) that if a Bayesian game is uniformly payoff secure, then it is payoff secure when viewed as a normal form game. Furthermore, the aggregate upper semicontinuity condition implies the corresponding property for the Bayesian game $\Gamma(v, \omega^k)$. Thus, one can apply the existence result in Reny (1999) to conclude the existence of equilibria. The second difficulty with this approach is that it is not clear to us whether the set of equilibrium payoffs in the Bayesian game $\Gamma(v, \omega^k)$ is convex or not. Due to lack of the convexity property, the fixed point argument is not readily applicable. To address the convexity issue, the works in the literature often impose various conditions on the state structure. In this paper, we do not adopt those restrictions.

We now start to sketch the proof of Theorem 1, which is divided into six steps.

Step 1. The mappings $\{W_i^k\}_{i \in I, 1 \le k \le K}$ are important for the analysis. Roughly, $W_i^k(f_i, f_{-i})$ is the payoff of player i at state ω^k when he plays the strategy f_i at the current stage, given that the other players always play the stationary strategies f_{-i}, and player i plays the optimal strategy (not necessarily f_i) in the future. In order to construct $W_i^k(f_i, f_{-i})$, we define in this step a mapping ϕ_i from $\mathcal{L}_{-i} \times \tilde{V}_i$ to $\tilde{V}_i$: given $f_{-i} \in \mathcal{L}_{-i}$ and $\tilde{v}_i \in \tilde{V}_i$, for $1 \le k \le K$,

$$\phi_i^k(f_{-i}, \tilde{v}_i) = \sup_{g_i \in \mathcal{L}_i} R_i^k(g_i, f_{-i}, \tilde{v}_i).$$

That is, $\phi_i^k(f_{-i}, \tilde{v}_i)$ is player i's maximally possible payoff at state ω^k, if other players adopt the stationary strategies f_{-i} and player i's continuation value is $\tilde{v}_i$. For each $f_{-i} \in \mathcal{L}_{-i}$, it is obvious that $\phi_i(f_{-i}, \cdot)$ is a contraction mapping on $\tilde{V}_i$. There exists a unique vector $\varphi_i(f_{-i}) \in \tilde{V}_i$ such that

$$\varphi_i(f_{-i}) = \phi_i(f_{-i}, \varphi_i(f_{-i})).$$

Based on the condition of continuation payoff security, it is shown that ϕ_i^k is lower semicontinuous for $i \in I$ and $1 \leq k \leq K$.

Step 2. Note that the payoff W_i may not be easy to work with, given the endogenous discontinuity of the mapping ϕ_i (and hence φ_i). To address this issue, we shall construct in Step 3 a sequence of well-behaved payoffs $\{W_{ij}\}_{j \geq 1}$ to approximate W_i. Towards this end, we first obtain an increasing sequence of continuous mappings $\{\phi_{ij}\}_{j \geq 1}$ from $\mathcal{L}_{-i} \times \tilde{V}_i$ to $\tilde{V}_i$ to approximate ϕ_i from below. Importantly, for each $j \geq 1$ and $f_{-i} \in \mathcal{L}_{-i}$, $\phi_{ij}(f_{-i}, \cdot)$ inherits the contraction property of $\phi_i(f_{-i}, \cdot)$ on $\tilde{V}_i$, and hence possesses a unique fixed point $\varphi_{ij}(f_{-i}) \in \tilde{V}_i$ such that φ_{ij} is continuous, and

$$\varphi_{ij}(f_{-i}) = \phi_{ij}(f_{-i}, \varphi_{ij}(f_{-i})).$$

Step 3. Now we are ready to define

$$W_{ij}(f_i, f_{-i}) = R_i(f_i, f_{-i}, \varphi_{ij}(f_{-i}))$$

for each $i \in I$, $j \geq 1$ and $f \in \mathcal{L}$. Based on the condition of continuation payoff security, $R_i^k(\cdot, \cdot, v_i)$ is payoff secure for $i \in I$, $1 \leq k \leq K$ and $v_i \in V_i$. As φ_{ij} is continuous for $j \geq 1$, W_{ij}^k, the composition of R_i^k and φ_{ij}, is shown to be also payoff secure.

Step 4. For $i \in I$, $j \geq 1$ and $f_{-i} \in \mathcal{L}_{-i}$, define

$$\Upsilon_{ij}(f_{-i}) = \left\{ g_i \in \mathcal{L}_i \colon W_{ij}^k(g_i, f_{-i}) \geq \varphi_{ij}^k(f_{-i}) - \frac{1}{j}, \, 1 \leq k \leq K \right\}.$$

It will be shown that the correspondence $\Upsilon_j = \prod_{i \in I} \Upsilon_{ij}$ possesses a fixed point $(f_{1j}^*, \ldots, f_{nj}^*) \in \mathcal{L}$ for each $j \geq 1$.

Step 5. Given the sequence $\{(f_{1j}^*, \ldots, f_{nj}^*)\}_{j \geq 1}$, one can obtain a limit $f^* = (f_1^*, \ldots, f_n^*) \in \mathcal{L}$ (possibly by focusing on a subsequence). As explained in the beginning of this subsection, the aim is to show that for $i \in I$ and $1 \leq k \leq K$,

$$\varphi_i^k(f_{-i}^*) = R_i^k(f_i^*, f_{-i}^*, \varphi_i(f_{-i}^*)) = W_i^k(f_i^*, f_{-i}^*),$$

and hence by the definition of φ_i^k, for any $g_i \in \mathcal{L}_i$,

$$W_i^k(f_i^*, f_{-i}^*) \geq W_i^k(g_i, f_{-i}^*).$$

To obtain such results, a common proof technique in the literature is to employ the condition of aggregate upper semicontinuity.[19] As we impose a similar condition for stochastic games, one can also show that $\sum_{i \in I} R_i^k(\cdot, \cdot, v_i)$ is upper semicontinuous given any $v \in V$, and hence hope that similar proof technique can be adopted here. However, new difficulty arises due to the dynamic nature of our problem. As mentioned above, φ_i^k is lower semicontinuous for $1 \leq k \leq K$, which makes it unclear whether $\sum_{i \in I} W_i^k(f_i, f_{-i}) = \sum_{i \in I} R_i^k(f_i, f_{-i}, \varphi_i(f_{-i}))$ is upper semicontinuous in (f_i, f_{-i}) or not. The key observation in this step is that even though we are not sure whether $\sum_{i \in I} W_i^k$ is upper semicontinuous in general, the following result holds for the special sequence $\{(f_{1j}^*, \ldots, f_{nj}^*)\}_{j \geq 1}$ and $(f_1^*, \ldots, f_n^*)$: for $i \in I$ and $1 \leq k \leq K$,

$$\limsup_{j \to \infty} \sum_{i \in I} W_{ij}^k(f_{ij}^*, f_{(-i)j}^*) \leq \sum_{i \in I} W_i^k(f_i^*, f_{-i}^*).$$

Step 6. The last step verifies that $(f_1^*, \ldots, f_n^*) \in \mathcal{L}$ is a stationary Markov perfect equilibrium in the stochastic game. Let $(v_1^*, \ldots, v_n^*)$ be the continuation payoff generated by $f^* = (f_1^*, \ldots, f_n^*)$ via the Bellman equation (1), and v_i^{k*} be the continuation payoff of player i when the common state is ω_k. For $i \in I$ and

[19] See, for example, Reny (1999), Prokopovych (2011), and Carbonell-Nicolau and McLean (2018).

$1 \leq k \leq K$, let

$$\tilde{v}_i^{k*} = \int_{S_i} v_i^{k*}(s_i) P_i(\mathrm{d}s_i|\omega^k).$$

As $\tilde{v}_i^*$ and $\varphi_i(f_{-i}^*)$ are both fixed points of $R_i(f_i^*, f_{-i}^*, \cdot)$ in $\tilde{V}_i$, $\tilde{v}_i^* = \varphi_i(f_{-i}^*)$. This implies that $f^* = (f_1^*, \ldots, f_n^*)$ is a stationary Markov perfect equilibrium.

6 Appendix

6.1 Proof of Theorem 1

The proof is divided into six steps.

Step 1. Recall that $\tilde{V}_i^k = [-\frac{C}{1-\beta_i}, \frac{C}{1-\beta_i}]$ for $1 \leq k \leq K$, and $\tilde{V}_i = \prod_{1 \leq k \leq K} \tilde{V}_i^k$. For $i \in I$, $1 \leq k \leq K$, $f = (f_i, f_{-i}) \in \mathcal{L}$, and $v_i \in V_i$,

$$R_i^k(f_i, f_{-i}, v_i) = \int_S \int_X U_i(\omega^k, \mathbf{s}, \mathbf{x}, v_i) \otimes_{m \in I} f_m(\mathrm{d}x_m|\omega^k, s_m) P(\mathrm{d}\mathbf{s}|\omega^k).$$

Given $f_{-i} \in \mathcal{L}_{-i}$ and $\tilde{v}_i \in \tilde{V}_i$, for $1 \leq k \leq K$,

$$\phi_i^k(f_{-i}, \tilde{v}_i) = \sup_{g_i \in \mathcal{L}_i} R_i^k(g_i, f_{-i}, \tilde{v}_i).$$

For $f_{-i} \in \mathcal{L}_{-i}$, it is clear that $\phi_i(f_{-i}, \cdot)$ is a contraction mapping on $\tilde{V}_i$. Then there exists a unique vector $\varphi_i(f_{-i}) \in \tilde{V}_i$ such that

$$\varphi_i(f_{-i}) = \phi_i(f_{-i}, \varphi_i(f_{-i})).$$

In this step, we show that ϕ_i^k is lower semicontinuous for $i \in I$ and $1 \leq k \leq K$.

Suppose that for some $i \in I$ and $1 \leq k \leq K$, ϕ_i^k is not lower semicontinuous. Then there exists some $\epsilon_1 > 0$ and a sequence $\{(f_{(-i)j}, \tilde{v}_{ij})\}_{j \geq 1} \subseteq \mathcal{L}_{-i} \times \tilde{V}_i$ with $(f_{(-i)j}, \tilde{v}_{ij}) \to (f_{(-i)0}, \tilde{v}_{i0})$ as $j \to \infty$, such that for $j \geq 1$,

$$\phi_i^k(f_{(-i)j}, \tilde{v}_{ij}) < \phi_i^k(f_{(-i)0}, \tilde{v}_{i0}) - \epsilon_1.$$

Then

$$\liminf_{j \geq 1} \phi_i^k(f_{(-i)j}, \tilde{v}_{ij}) - \phi_i^k(f_{(-i)0}, \tilde{v}_{i0}) \leq -\epsilon_1.$$

Since $\tilde{v}_{ij} \to \tilde{v}_{i0}$, there exists some sufficiently large integer $N_1 > 0$ such that $\tilde{v}_{i0}^{\nu} - \tilde{v}_{ij}^{\nu} < \epsilon_1$ for each $1 \leq \nu \leq K$ and $j \geq N_1$. Then for $j \geq N_1$,

$$\begin{aligned}
&\phi_i^k \left(f_{(-i)j}, \tilde{v}_{ij}\right) - \phi_i^k \left(f_{(-i)j}, \tilde{v}_{i0}\right) \\
&= \sup_{g_i \in \mathcal{L}_i} \int_S \int_X \Bigg[u_i(\omega^k, s_i, s_{-i}, x_i, x_{-i}) + \\
&\quad \beta_i \sum_{1 \leq l \leq K} \tilde{v}_{ij}^l Q(\omega^l | \omega^k, x_i, x_{-i})\Bigg] f_{(-i)j}(\mathrm{d}x_{-i} | \omega^k, s_{-i}) \otimes g_i(\mathrm{d}x_i | \omega^k, s_i) P(\mathrm{d}\mathbf{s} | \omega^k) \\
&- \sup_{g_i \in \mathcal{L}_i} \int_S \int_X \Bigg[u_i(\omega^k, s_i, s_{-i}, x_i, x_{-i}) + \\
&\quad \beta_i \sum_{1 \leq l \leq K} \tilde{v}_{i0}^l Q(\omega^l | \omega^k, x_i, x_{-i})\Bigg] f_{(-i)j}(\mathrm{d}x_{-i} | \omega^k, s_{-i}) \otimes g_i(\mathrm{d}x_i | \omega^k, s_i) P(\mathrm{d}\mathbf{s} | \omega^k) \\
&\geq \inf_{g_i \in \mathcal{L}_i} \beta_i \int_S \int_X \sum_{1 \leq l \leq K} \left[\tilde{v}_{ij}^l - \tilde{v}_{i0}^l\right] Q(\omega^l | \omega^k, x_i, x_{-i}) \\
&\qquad f_{(-i)j}(\mathrm{d}x_{-i} | \omega^k, s_{-i}) \otimes g_i(\mathrm{d}x_i | \omega^k, s_i) P(\mathrm{d}\mathbf{s} | \omega^k) \\
&> -\beta_i \epsilon_1.
\end{aligned} \tag{3}$$

By the definition of ϕ_i^k, there exists some $g_i \in \mathcal{L}_i$ such that

$$R_i^k(g_i, f_{(-i)0}, \tilde{v}_{i0}) > \phi_i^k \left(f_{(-i)0}, \tilde{v}_{i0}\right) - \frac{1 - \beta_i}{2} \epsilon_1.$$

Viewing the Bayesian game $\Gamma(\tilde{v}_0, \omega^k)$ as a normal-form game with action space $\mathcal{L}_i^k$ and payoff $R_i^k(\cdot, \cdot, \tilde{v}_{i0})$ for player i, by Lemma 2 in Carbonell-Nicolau and McLean (2018), the game is payoff secure. Thus, one can find some $g_i' \in \mathcal{L}_i$ such that

$$R_i^k(g_i', f_{(-i)0}', \tilde{v}_{i0}) > \phi_i^k \left(f_{(-i)0}, \tilde{v}_{i0}\right) - \frac{1 - \beta_i}{2} \epsilon_1$$

for any $f_{(-i)0}'$ in some neighborhood $O_{f_{(-i)0}}$ of $f_{(-i)0}$. There exists some positive integer $\overline{N}_1 \geq N_1$ such that $f_{(-i)j} \in O_{f_{(-i)0}}$ for $j \geq \overline{N}_1$. As a result,

$$\phi_i^k \left(f_{(-i)j}, \tilde{v}_{i0}\right) \geq R_i^k(g_i', f_{(-i)j}, \tilde{v}_{i0}) > \phi_i^k \left(f_{(-i)0}, \tilde{v}_{i0}\right) - \frac{1 - \beta_i}{2} \epsilon_1. \tag{4}$$

Combining (3) and (4) above, we have that for $j \geq \overline{N}_1$,

$$\begin{aligned}
\phi_i^k\left(f_{(-i)j}, \tilde{v}_{ij}\right) - \phi_i^k\left(f_{(-i)0}, \tilde{v}_{i0}\right) &= \phi_i^k\left(f_{(-i)j}, \tilde{v}_{ij}\right) - \phi_i^k\left(f_{(-i)j}, \tilde{v}_{i0}\right) \\
&\quad + \phi_i^k\left(f_{(-i)j}, \tilde{v}_{i0}\right) - \phi_i^k\left(f_{(-i)0}, \tilde{v}_{i0}\right) \\
&> -\beta_i \epsilon_1 - \frac{1-\beta_i}{2}\epsilon_1 \\
&= -\frac{1+\beta_i}{2}\epsilon_1,
\end{aligned}$$

which implies that

$$-\epsilon_1 \geq \liminf_{j\to\infty} \phi_i^k\left(f_{(-i)j}, \tilde{v}_{ij}\right) - \phi_i^k\left(f_{(-i)0}, \tilde{v}_{i0}\right) \geq -\frac{1+\beta_i}{2}\epsilon_1 > -\epsilon_1.$$

This is a contradiction. Therefore, ϕ_i^k is lower semicontinuous for each $i \in I$ and $1 \leq k \leq K$.

Step 2. Following the argument in the first paragraph of Sun (1997, p.149), the topology of weak convergence on $\mathcal{L}_i$ is metrizable for each $i \in I$. Let d_{-i} be the metric on $\mathcal{L}_{-i}$. In this step, we construct a sequence of mappings $\{\phi_{ij}\}_{j\geq 1}$ from $\mathcal{L}_{-i} \times \tilde{V}_i$ to $\tilde{V}_i$ as follows: for $1 \leq k \leq K$, $j \geq 1$, $f_{-i} \in \mathcal{L}_{-i}$ and $\tilde{v}_i \in \tilde{V}_i$,

$$\phi_{ij}^k(f_{-i}, \tilde{v}_i) = \inf_{g_{-i} \in \mathcal{L}_{-i}} \left\{\phi_i^k(g_{-i}, \tilde{v}_i) + j \cdot d_{-i}(f_{-i}, g_{-i})\right\}.$$

We show that

1. for $1 \leq k \leq K$, $\{\phi_{ij}^k\}_{j\geq 1}$ is an increasing sequence of continuous mappings;
2. for each $j \geq 1$ and $f_{-i} \in \mathcal{L}_{-i}$, $\phi_{ij}(f_{-i}, \cdot)$ is a contraction mapping on $\tilde{V}_i$;
3. for $1 \leq k \leq K$ and each $(f_{-i}, \tilde{v}_i) \in \mathcal{L}_{-i} \times \tilde{V}_i$,

$$\lim_{j\to\infty} \phi_{ij}^k(f_{-i}, \tilde{v}_i) = \phi_i^k(f_{-i}, \tilde{v}_i).$$

For each $j \geq 1$ and $f_{-i} \in \mathcal{L}_{-i}$, since $\phi_{ij}(f_{-i}, \cdot)$ is a contraction mapping, there exists a unique $\varphi_{ij}(f_{-i}) \in \tilde{V}_i$ such that

$$\phi_{ij}(f_{-i}, \varphi_{ij}(f_{-i})) = \varphi_{ij}(f_{-i}).$$

It will be clear that

a. φ_{ij} is continuous;

b. for $1 \leq k \leq K$ and any sequence $\{f_{(-i)j}\}_{j\geq 1} \subseteq \mathcal{L}_{-i}$ with $f_{(-i)j} \to f_{(-i)0}$,

$$\liminf_{j\to\infty} \varphi_{ij}^k(f_{(-i)j}) \geq \varphi_i^k(f_{(-i)0}).$$

We first prove properties (1)-(3) for the sequence $\{\phi_{ij}\}_{j\geq 1}$. It is clear that for $1 \leq k \leq K$,

$$\phi_{ij}^k(f_{-i}, \tilde{v}_i) \leq \phi_{i(j+1)}^k(f_{-i}, \tilde{v}_i) \leq \phi_i^k(f_{-i}, \tilde{v}_i).$$

For any $f_{-i}, f'_{-i} \in \mathcal{L}_{-i}$ and $\tilde{v}_i, \tilde{v}'_i \in \tilde{V}_i$,

$$|\phi_{ij}^k(f_{-i}, \tilde{v}_i) - \phi_{ij}^k(f'_{-i}, \tilde{v}'_i)| \leq j \cdot d_{-i}(f_{-i}, f'_{-i}) + \beta_i \|\tilde{v}_i - \tilde{v}'_i\|.$$

Thus, ϕ_{ij}^k is Lipschitz continuous, and $\phi_{ij}(f_{-i}, \cdot)$ is a contraction mapping on $\tilde{V}_i$.

Fix $(f_{-i}, \tilde{v}_i) \in \mathcal{L}_{-i} \times \tilde{V}_i$ and some $\epsilon_2 > 0$. Suppose that $\phi_{ij}^k(f_{-i}, \tilde{v}_i) \to \alpha$ as $j \to \infty$. Then $\alpha \leq \phi_i^k(f_{-i}, \tilde{v}_i)$. For each $j \geq 1$, pick $g_{(-i)j}$ such that

$$\phi_i^k(g_{(-i)j}, \tilde{v}_i) \leq \phi_i^k(g_{(-i)j}, \tilde{v}_i) + j \cdot d_{-i}(f_{-i}, g_{(-i)j}) < \phi_{ij}^k(f_{-i}, \tilde{v}_i) + \epsilon_2.$$

Thus, as $j \to \infty$,

$$d_{-i}(f_{-i}, g_{(-i)j}) \leq \frac{\phi_{ij}^k(f_{-i}, \tilde{v}_i) + \epsilon_2 - \phi_i^k(g_{(-i)j}, \tilde{v}_i)}{j} < \frac{\phi_i^k(f_{-i}, \tilde{v}_i) + \epsilon_2}{j} \to 0,$$

which implies that $g_{(-i)j} \to f_{-i}$ as $j \to \infty$. Since ϕ_i^k is lower semicontinuous,

$$\phi_i^k(f_{-i}, \tilde{v}_i) \leq \liminf_{j\to\infty} \phi_i^k(g_{(-i)j}, \tilde{v}_i) \leq \liminf_{j\to\infty} [\phi_{ij}^k(f_{-i}, \tilde{v}_i) + \epsilon_2] = \alpha + \epsilon_2.$$

As ϵ_2 is arbitrary,

$$\phi_i^k(f_{-i}, \tilde{v}_i) = \alpha = \lim_{j\to\infty} \phi_{ij}^k(f_{-i}, \tilde{v}_i).$$

Next, we consider properties (a) and (b) of the sequence $\{\varphi_{ij}\}_{j\geq 1}$. We first prove property (a). Fix $j \geq 1$ and a sequence $\{f_{(-i)l}\}_{l\geq 1} \subseteq \mathcal{L}_{-i}$ such that $f_{(-i)l} \to f_{(-i)0}$ as $l \to \infty$. Pick any subsequence of $\{f_{(-i)l}\}_{l\geq 1}$ such that $\{\varphi_{ij}(f_{(-i)l})\}_{l\geq 1}$ is convergent to some $\alpha_0 = (\alpha_1, \dots, \alpha_K) \in \tilde{V}_i$ as $l \to \infty$. Since ϕ_{ij} is continuous, as $l \to \infty$,

$$\varphi_{ij}(f_{(-i)l}) = \phi_{ij}\left(f_{(-i)l}, \varphi_{ij}(f_{(-i)l})\right) \to \phi_{ij}\left(f_{(-i)0}, \alpha_0\right),$$

which implies that

$$\phi_{ij}\left(f_{(-i)0}, \alpha_0\right) = \alpha_0.$$

Since the fixed point of $\phi_{ij}(f_{(-i)0}, \cdot)$ is unique, $\alpha_0 = \varphi_{ij}(f_{(-i)0})$. Thus, φ_{ij} is continuous.

Now consider property (b). Suppose that this property is not true. Then there is some $\tilde{\epsilon}_2 > 0$, $1 \le k \le K$, and a sequence $\{f_{(-i)j}\}_{j\ge 1} \subseteq \mathcal{L}_{-i}$ with $f_{(-i)j} \to f_{(-i)0}$,

$$\lim_{j\to\infty} \varphi_{ij}^k(f_{(-i)j}) = \varphi_i^k(f_{(-i)0}) - \tilde{\epsilon}_2.$$

Without loss of generality, assume that for $1 \le \nu \le K$,

$$\lim_{j\to\infty} \varphi_{ij}^\nu(f_{(-i)j}) - \varphi_i^\nu(f_{(-i)0}) \ge \lim_{j\to\infty} \varphi_{ij}^k(f_{(-i)j}) - \varphi_i^k(f_{(-i)0}) = -\tilde{\epsilon}_2.$$

There exists some sufficiently large N_2 such that for $1 \le \nu \le K$ and $j \ge N_2$,

$$\varphi_{ij}^\nu(f_{(-i)j}) - \varphi_i^\nu(f_{(-i)0}) > -\frac{1+\beta_i}{2\beta_i}\tilde{\epsilon}_2.$$

Then we have that for $j \ge N_2$,

$$\begin{aligned}
&\phi_{ij}^k(f_{(-i)j}, \varphi_{ij}(f_{(-i)j})) - \phi_{ij}^k(f_{(-i)j}, \varphi_i(f_{(-i)0})) \\
&= \sup_{g_i\in\mathcal{L}_i} \int_S \int_X \Bigg[u_i(\omega^k, s_i, s_{-i}, x_i, x_{-i}) + \\
&\beta_i \sum_{1\le l\le K} \varphi_{ij}^l(f_{(-i)j}) Q(\omega^l|\omega^k, x_i, x_{-i})\Bigg] f_{(-i)j}(\mathrm{d}x_{-i}|\omega^k, s_{-i}) \otimes g_i(\mathrm{d}x_i|\omega^k, s_i) P(\mathrm{d}\mathbf{s}|\omega^k) \\
&- \sup_{g_i\in\mathcal{L}_i} \int_S \int_X \Bigg[u_i(\omega^k, s_i, s_{-i}, x_i, x_{-i}) + \\
&\beta_i \sum_{1\le l\le K} \varphi_i^l(f_{(-i)0}) Q(\omega^l|\omega^k, x_i, x_{-i})\Bigg] f_{(-i)j}(\mathrm{d}x_{-i}|\omega^k, s_{-i}) \otimes g_i(\mathrm{d}x_i|\omega^k, s_i) P(\mathrm{d}\mathbf{s}|\omega^k) \\
&\ge \inf_{g_i\in\mathcal{L}_i} \beta_i \int_S \int_X \sum_{1\le l\le K} \left[\varphi_{ij}^l(f_{(-i)j}) - \varphi_i^l(f_{(-i)0})\right] Q(\omega^l|\omega^k, x_i, x_{-i})
\end{aligned}$$

$$f_{(-i)j}(\mathrm{d}x_{-i}|\omega^k, s_{-i}) \otimes g_i(\mathrm{d}x_i|\omega^k, s_i)P(\mathrm{d}\mathbf{s}|\omega^k)$$
$$> -\frac{1+\beta_i}{2}\tilde{\epsilon}_2.$$

Furthermore, we claim that

$$\liminf_{j\geq\infty} \phi_{ij}^k(f_{(-i)j}, \varphi_i(f_{(-i)0})) \geq \phi_i^k(f_{(-i)0}, \varphi_i(f_{(-i)0})).$$

If it is not true, then there is a subsequence, say itself, such that for some $\hat{\epsilon}_2 > 0$,

$$\lim_{j\geq\infty} \phi_{ij}^k(f_{(-i)j}, \varphi_i(f_{(-i)0})) < \phi_i^k(f_{(-i)0}, \varphi_i(f_{(-i)0})) - \hat{\epsilon}_2.$$

As $\phi_{ij}^k(f_{(-i)0}, \varphi_i(f_{(-i)0})) \to \phi_i^k(f_{(-i)0}, \varphi_i(f_{(-i)0}))$, there is a sufficiently large integer $\hat{N}_2$ such that

$$\lim_{j\geq\infty} \phi_{ij}^k(f_{(-i)j}, \varphi_i(f_{(-i)0})) < \phi_{i\hat{N}_2}^k(f_{(-i)0}, \varphi_i(f_{(-i)0})) - \hat{\epsilon}_2.$$

Since $\phi_{i\hat{N}_2}^k$ is continuous, there exists some $\overline{N}_2 \geq \hat{N}_2$ such that for $j \geq \overline{N}_2$,

$$\begin{aligned}\lim_{j\geq\infty} \phi_{ij}^k(f_{(-i)j}, \varphi_i(f_{(-i)0})) &< \phi_{i\hat{N}_2}^k(f_{(-i)j}, \varphi_i(f_{(-i)0})) - \hat{\epsilon}_2 \\ &\leq \phi_{ij}^k(f_{(-i)j}, \varphi_i(f_{(-i)0})) - \hat{\epsilon}_2.\end{aligned}$$

The second inequality follows from the fact that $\{\phi_{ij}^k\}_{j\geq 1}$ is an increasing sequence. Taking j to infinity on the right hand side of the inequality above, we arrive at a contradiction. This proves the claim. Therefore, there exists some sufficiently large integer, say N_2, such that for $j \geq N_2$,

$$\phi_{ij}^k(f_{(-i)j}, \varphi_i(f_{(-i)0})) - \phi_i^k(f_{(-i)0}, \varphi_i(f_{(-i)0})) > -\frac{1-\beta_i}{4}\tilde{\epsilon}_2.$$

Then we have that for $j \geq N_2$,

$$\begin{aligned}\varphi_{ij}^k(f_{(-i)j}) - \varphi_i^k(f_{(-i)0}) &= \phi_{ij}^k(f_{(-i)j}, \varphi_{ij}(f_{(-i)j})) - \phi_i^k(f_{(-i)0}, \varphi_i(f_{(-i)0})) \\ &= \phi_{ij}^k(f_{(-i)j}, \varphi_{ij}(f_{(-i)j})) - \phi_{ij}^k(f_{(-i)j}, \varphi_i(f_{(-i)0})) \\ &\quad + \phi_{ij}^k(f_{(-i)j}, \varphi_i(f_{(-i)0})) - \phi_i^k(f_{(-i)0}, \varphi_i(f_{(-i)0}))\end{aligned}$$

$$
\begin{aligned}
&> -\frac{1+\beta_i}{2}\tilde{\epsilon}_2 - \frac{1-\beta_i}{4}\tilde{\epsilon}_2 \\
&= -\frac{3+\beta_i}{4}\tilde{\epsilon}_2.
\end{aligned}
$$

Taking j to infinity, we get $-\tilde{\epsilon}_2 \geq -\frac{3+\beta_i}{4}\tilde{\epsilon}_2$, which is a contradiction. This proves property (b).

Step 3. For $i \in I$, $j \geq 1$ and $f \in \mathcal{L}$, let

$$
W_{ij}(f_i, f_{-i}) = R_i(f_i, f_{-i}, \varphi_{ij}(f_{-i}))
$$

and

$$
W_i(f_i, f_{-i}) = R_i(f_i, f_{-i}, \varphi_i(f_{-i})).
$$

Fix $j \geq 1$. In the followings, we show that W_{ij}^k is payoff secure for $1 \leq k \leq K$.

Fix $\epsilon_3 > 0$. Given $f = (f_i, f_{-i}) \in \mathcal{L}$ and $1 \leq k \leq K$, by the payoff security of the Bayesian game $\Gamma(\varphi_{ij}(f_{-i}), \omega^k)$, there exists some $g_i \in \mathcal{L}_i$ such that

$$
\begin{aligned}
R_i^k(g_i, f'_{-i}, \varphi_{ij}(f_{-i})) &> R_i^k(f_i, f_{-i}, \varphi_{ij}(f_{-i})) - \frac{\epsilon_3}{2} \\
&= W_{ij}^k(f_i, f_{-i}) - \frac{\epsilon_3}{2}
\end{aligned}
$$

for any f'_{-i} in some neighborhood $O_{f_{-i}}$ of f_{-i}.

Since φ_{ij} is continuous, there exists a neighborhood $O'_{f_{-i}}$ of f_{-i} such that $O'_{f_{-i}} \subseteq O_{f_{-i}}$, and

$$
\varphi_{ij}^{\nu}(f'_{-i}) - \varphi_{ij}^{\nu}(f_{-i}) > -\frac{\epsilon_3}{2}
$$

for any $f'_{-i} \in O'_{f_{-i}}$ and $1 \leq \nu \leq K$. Then we have

$$
\begin{aligned}
& W_{ij}^k(g_i, f'_{-i}) - R_i^k(g_i, f'_{-i}, \varphi_{ij}(f_{-i})) \\
&= R_i^k(g_i, f'_{-i}, \varphi_{ij}(f'_{-i})) - R_i^k(g_i, f'_{-i}, \varphi_{ij}(f_{-i})) \\
&= \int_S \int_X \Bigg[\quad \beta_i \sum_{1 \leq \nu \leq K} [\varphi_{ij}^{\nu}(f'_{-i}) - \varphi_{ij}^{\nu}(f_{-i})] Q(\omega^{\nu} | \omega^k, x_i, x_{-i}) \Bigg] \\
&\quad \otimes_{m \neq i} f'_m(\mathrm{d}x_m | \omega^k, s_m) \otimes g_i(\mathrm{d}x_i | \omega^k, s_i) P(\mathrm{d}\mathbf{s} | \omega^k) \\
&> -\frac{\epsilon_3}{2}.
\end{aligned}
$$

Therefore, for any $f'_{-i} \in O'_{f_{-i}}$,

$$W_{ij}^k(g_i, f'_{-i}) - W_{ij}^k(f_i, f_{-i}) > -\epsilon_3.$$

It implies that W_{ij}^k is payoff secure.

Step 4. For each $i \in I$ and $j \geq 1$, recall that

$$\Upsilon_{ij}(f_{-i}) = \left\{ g_i \in \mathcal{L}_i \colon W_{ij}^k(g_i, f_{-i}) \geq \varphi_{ij}^k(f_{-i}) - \frac{1}{j},\, 1 \leq k \leq K \right\}.$$

It is clear that for $i \in I$, $f_{-i} \in \mathcal{L}_{-i}$, $1 \leq k \leq K$ and $j \geq 1$,

$$\phi_i^k\left(f_{-i}, \varphi_{ij}(f_{-i})\right) \geq \phi_{ij}^k\left(f_{-i}, \varphi_{ij}(f_{-i})\right) = \varphi_{ij}^k(f_{-i}).$$

Thus, Υ_{ij} is nonempty valued. It is clear that Υ_{ij} is convex valued. Then $\Upsilon_j = \prod_{i \in I} \Upsilon_{ij}$ is a nonempty and convex valued correspondence from $\mathcal{L}$ to itself. We shall show that for each $j \geq 1$, the correspondence Υ_j possesses a fixed point $(f_{1j}^*, \ldots, f_{nj}^*) \in \mathcal{L}$.

Fix $j \geq 1$. For any $f \in \mathcal{L}$ and $i \in I$, there exists some $f'_i \in \mathcal{L}_i$ such that for $1 \leq k \leq K$,

$$W_{ij}^k(f'_i, f_{-i}) > \varphi_{ij}^k(f_{-i}) - \frac{1}{2j}.$$

Since W_{ij}^k is payoff secure and φ_{ij}^k is continuous, $W_{ij}^k - \varphi_{ij}^k$ is also payoff secure. There exists some $g_i \in \mathcal{L}_i$ and a neighborhood $O_{f_{-i}}$ of f_{-i} such that

$$W_{ij}^k(g_i, f'_{-i}) - \varphi_{ij}^k(f'_{-i}) > W_{ij}^k(f'_i, f_{-i}) - \varphi_{ij}^k(f_{-i}) - \frac{1}{2j} > -\frac{1}{j} \tag{5}$$

for any $f'_{-i} \in O_{f_{-i}}$ and $1 \leq k \leq K$. By the definition of Υ_{ij}, $g_i \in \Upsilon_{ij}(f'_{-i})$ for any $f'_{-i} \in O_{f_{-i}}$.

Let $O_f = \cap_{i \in I}(\mathcal{L}_i \times O_{f_{-i}})$, which is an open neighborhood of f. The collection $\mathscr{C} = \{O_f \colon f \in \mathcal{L}\}$ is an open cover of $\mathcal{L}$. Since $\mathcal{L}$ is weak* compact, there is a finite set $\{f^1, \ldots, f^L\}$ such that $\mathcal{L} \subseteq \cup_{1 \leq l \leq L} O_{f^l}$. Let $\{E_{f^l}\}_{1 \leq l \leq L}$ be a closed refinement; that is, $E_{f^l} \subseteq O_{f^l}$, E_{f^l} is weak* closed and $\mathcal{L} = \cup_{1 \leq l \leq L} E_{f^l}$; see Michael (1953, Lemma 1). Let $\{g_i^l\}_{1 \leq l \leq L}$ be such that g_i^l is the mapping given by (5) for player i

and f^l, $1 \le l \le L$.

For each $f \in \mathcal{L}$, let

$$\Lambda(f) = \{1 \le l \le L \colon f \in E_{f^l}\}, \qquad \text{and } \Psi_i(f) = \mathrm{co}\left(\cup_{l \in \Lambda(f)} \{g_i^l\}\right).$$

It is obvious that Ψ_i is nonempty and convex valued, and $\Psi_i(f) \subseteq \Upsilon_{ij}(f)$ for each $f \in \mathcal{L}$. By Aliprantis and Border (2006, Lemma 5.29), Ψ_i is weak* compact valued. Denote $\psi_i^l(f) = \{g_i^l\}$ if $f \in E_{f^l}$, and $\emptyset$ otherwise. Then ψ_i^l is an upper hemicontinuous correspondence on $\mathcal{L}$. The correspondence $\cup_{1 \le l \le L} \psi_i^l(f)$ is the union of finitely many upper hemicontinuous correspondences, and hence is upper hemicontinuous; see Aliprantis and Border (2006, Theorem 17.57). Since $\Psi_i(f) = \mathrm{co}\left(\cup_{1 \le l \le L} \psi_i^l(f)\right)$ and is compact valued, it is also upper hemicontinuous; see Aliprantis and Border (2006, Theorem 17.35). Then the correspondence $\Psi = \prod_{i \in I} \Psi_i$ is nonempty, convex and compact valued, and upper hemicontinuous. By Fan-Glicksberg's fixed-point theorem, there exists a point $(f_{1j}^*, \ldots, f_{nj}^*) \in \mathcal{L}$ such that

$$(f_{1j}^*, \ldots, f_{nj}^*) \in \Psi(f_{1j}^*, \ldots, f_{nj}^*) \subseteq \Upsilon_j(f_{1j}^*, \ldots, f_{nj}^*).$$

Step 5. Since $\mathcal{L}$ is weak* compact, there exists a subsequence of $\{(f_{1j}^*, \ldots, f_{nj}^*)\}_{j \ge 1}$, say itself, which converges to some $f^* = (f_1^*, \ldots, f_n^*) \in \mathcal{L}$. By further focusing on a subsequence,, we assume that $\{\varphi_{ij}^k(f_{(-i)j}^*)\}_{j \ge 1}$ is convergent for each $1 \le k \le K$ and $i \in I$. We shall show that for $1 \le k \le K$,

$$\limsup_{j \to \infty} \sum_{i \in I} W_{ij}^k(f_{ij}^*, f_{(-i)j}^*) \le \sum_{i \in I} W_i^k(f_i^*, f_{-i}^*).$$

In order to show this inequality, we need to first prove that

$$\lim_{j \to \infty} \sum_{i \in I} \varphi_{ij}^k(f_{(-i)j}^*) = \sum_{i \in I} \varphi_i^k(f_{-i}^*).$$

Suppose that this is not true. Recall that

$$\lim_{j \to \infty} \varphi_{ij}^k(f_{(-i)j}^*) = \liminf_{j \to \infty} \varphi_{ij}^k(f_{(-i)j}^*) \ge \varphi_i^k(f_{-i}^*)$$

for each $i \in I$ and $1 \leq k \leq K$. There exists some $\epsilon_5 > 0$, $k \in \{1, \ldots, K\}$, and a subsequence of $\{(f_{1j}^*, \ldots, f_{nj}^*)\}_{j \geq 1}$, say itself, such that

$$\lim_{j \to \infty} \sum_{i \in I} \varphi_{ij}^k(f_{(-i)j}^*) = \sum_{i \in I} \varphi_i^k(f_{-i}^*) + \epsilon_5.$$

Without loss of generality, assume that for $1 \leq \nu \leq K$,

$$\lim_{j \to \infty} \sum_{i \in I} \varphi_{ij}^\nu(f_{(-i)j}^*) - \sum_{i \in I} \varphi_i^\nu(f_{-i}^*) \leq \lim_{j \to \infty} \sum_{i \in I} \varphi_{ij}^k(f_{(-i)j}^*) - \sum_{i \in I} \varphi_i^k(f_{-i}^*).$$

There exists some sufficiently large integer N_5 such that for $j \geq N_5$,

$$\begin{aligned} \sum_{i \in I} W_{ij}^k(f_{ij}^*, f_{(-i)j}^*) - \sum_{i \in I} W_i^k(f_i^*, f_{-i}^*) &\geq \sum_{i \in I} \varphi_{ij}^k(f_{(-i)j}^*) - \sum_{i \in I} \varphi_i^k(f_{-i}^*) - \frac{n}{j} \\ &> \frac{2 + \max_{i \in I}\{\beta_i\}}{3} \epsilon_5. \end{aligned} \tag{6}$$

The first inequality holds since $(f_{1j}^*, \ldots, f_{nj}^*)$ is a fixed point of Υ_j, and $\varphi_i^k(f_{-i}^*) \geq W_i^k(f_i^*, f_{-i}^*)$ for each $i \in I$. The second inequality follows from the choice of ϵ_5.

Because of the aggregate upper semicontinuity, there exists some sufficiently large integer $\hat{N}_5 \geq N_5$ such that for $j \geq \hat{N}_5$,

$$\sum_{i \in I} R_i^k(f_{ij}^*, f_{(-i)j}^*, \varphi_i(f_{-i}^*)) - \sum_{i \in I} R_i^k(f_i^*, f_{-i}^*, \varphi_i(f_{-i}^*)) < \frac{1 - \max_{i \in I}\{\beta_i\}}{6} \epsilon_5. \tag{7}$$

Furthermore, fix sufficiently small $\hat{\epsilon}_5 > 0$ such that

$$\max\{\beta_i\}_{i \in I} \cdot [(1 + \hat{\epsilon}_5)\epsilon_5 + 2\hat{\epsilon}_5] < \frac{1 + \max\{\beta_i\}_{i \in I}}{2} \epsilon_5.$$

Then there exists some sufficiently large integer $\tilde{N}_5 \geq \hat{N}_5$ such that for $j \geq \tilde{N}_5$ and $1 \leq l \leq K$,

i. for each $i \in I$,

$$\varphi_{ij}^l(f_{(-i)j}^*) - \varphi_i^l(f_{-i}^*) + \frac{\hat{\epsilon}_5}{n} \geq 0,$$

ii.

$$\sum_{i \in I} \varphi_{ij}^l(f_{(-i)j}^*) - \sum_{i \in I} \varphi_i^l(f_{-i}^*) \leq \sum_{i \in I} \varphi_{ij}^k(f_{(-i)j}^*) - \sum_{i \in I} \varphi_i^k(f_{-i}^*) + \hat{\epsilon}_5,$$

iii. and

$$\sum_{i\in I}\varphi_{ij}^k(f^*_{(-i)j}) - \sum_{i\in I}\varphi_i^k(f^*_{-i}) \le (1+\hat{\epsilon}_5)\epsilon_5.$$

Claim (i) follows from $\lim_{j\to\infty}\varphi_{ij}^l(f^*_{(-i)j}) \ge \varphi_i^l(f^*_{-i})$ for $i\in I$ and $1\le l\le K$. Claim (ii) holds since $\lim_{j\to\infty}\sum_{i\in I}\varphi_{ij}^k(f^*_{(-i)j}) - \sum_{i\in I}\varphi_i^k(f^*_{-i})$ is the largest among $\left\{\lim_{j\to\infty}\sum_{i\in I}\varphi_{ij}^l(f^*_{(-i)j}) - \sum_{i\in I}\varphi_i^l(f^*_{-i})\right\}_{1\le l\le K}$. Claim (iii) is due to the definition of ϵ_5.

For $j\ge \tilde{N}_5$,

$$\begin{aligned}
&\sum_{i\in I}R_i^k(f^*_{ij}, f^*_{(-i)j}, \varphi_{ij}(f^*_{(-i)j})) - \sum_{i\in I}R_i^k(f^*_{ij}, f^*_{(-i)j}, \varphi_i(f^*_{-i}))\\
&=\sum_{i\in I}\int_S\int_X\Big[u_i(\omega^k, s_i, s_{-i}, x_i, x_{-i})+\\
&\beta_i\sum_{1\le l\le K}\varphi_{ij}^l(f^*_{(-i)j})Q(\omega^l|\omega^k, x_i, x_{-i})\Big]\otimes_{m\in I}f^*_{mj}(\mathrm{d}x_m|\omega^k, s_m)P(\mathrm{d}\mathbf{s}|\omega^k)\\
&-\sum_{i\in I}\int_S\int_X\Big[u_i(\omega^k, s_i, s_{-i}, x_i, x_{-i})+\\
&\beta_i\sum_{1\le l\le K}\varphi_i^l(f^*_{-i})Q(\omega^l|\omega^k, x_i, x_{-i})\Big]\otimes_{m\in I}f^*_{mj}(\mathrm{d}x_m|\omega^k, s_m)P(\mathrm{d}\mathbf{s}|\omega^k)\\
&=\sum_{i\in I}\beta_i\int_S\int_X\sum_{1\le l\le K}\left[\varphi_{ij}^l(f^*_{(-i)j}) - \varphi_i^l(f^*_{-i})\right]Q(\omega^l|\omega^k, x_i, x_{-i})\\
&\qquad\otimes_{m\in I}f^*_{mj}(\mathrm{d}x_m|\omega^k, s_m)P(\mathrm{d}\mathbf{s}|\omega^k)\\
&<\sum_{i\in I}\beta_i\int_S\int_X\sum_{1\le l\le K}\left[\varphi_{ij}^l(f^*_{(-i)j}) - \varphi_i^l(f^*_{-i}) + \frac{\hat{\epsilon}_5}{n}\right]Q(\omega^l|\omega^k, x_i, x_{-i})\\
&\qquad\otimes_{m\in I}f^*_{mj}(\mathrm{d}x_m|\omega^k, s_m)P(\mathrm{d}\mathbf{s}|\omega^k)\\
&\le\max\{\beta_i\}_{i\in I}\int_S\int_X\sum_{1\le l\le K}\left[\sum_{i\in I}\varphi_{ij}^l(f^*_{(-i)j}) - \sum_{i\in I}\varphi_i^l(f^*_{-i}) + \hat{\epsilon}_5\right]Q(\omega^l|\omega^k, x_i, x_{-i})\\
&\qquad\otimes_{m\in I}f^*_{mj}(\mathrm{d}x_m|\omega^k, s_m)P(\mathrm{d}\mathbf{s}|\omega^k)\\
&\le\max\{\beta_i\}_{i\in I}\int_S\int_X\sum_{1\le l\le K}\left[\sum_{i\in I}\varphi_{ij}^k(f^*_{(-i)j}) - \sum_{i\in I}\varphi_i^k(f^*_{-i}) + 2\hat{\epsilon}_5\right]Q(\omega^l|\omega^k, x_i, x_{-i})\\
&\qquad\otimes_{m\in I}f^*_{mj}(\mathrm{d}x_m|\omega^k, s_m)P(\mathrm{d}\mathbf{s}|\omega^k)\\
&=\max\{\beta_i\}_{i\in I}\left[\sum_{i\in I}\varphi_{ij}^k(f^*_{(-i)j}) - \sum_{i\in I}\varphi_i^k(f^*_{-i}) + 2\hat{\epsilon}_5\right]
\end{aligned}$$

$$
\begin{aligned}
&\le \max\{\beta_i\}_{i\in I}\left[(1+\hat{\epsilon}_5)\epsilon_5 + 2\hat{\epsilon}_5\right] \\
&< \frac{1+\max\{\beta_i\}_{i\in I}}{2}\epsilon_5. \qquad (8)
\end{aligned}
$$

The first inequality holds since we simply add a positive number $\frac{\hat{\epsilon}_5}{n}$. The second inequality is due to Claim (i) above and $\beta_l \le \max\{\beta_i\}_{i\in I}$ for each l. The third and fourth inequalities are true because of Claims (ii) and (iii), respectively. The last inequality is due to the choice of $\hat{\epsilon}_5$.

By (7) and (8) above, for $j \ge \tilde{N}_5$,

$$
\begin{aligned}
&\sum_{i\in I} W_{ij}^k(f_{ij}^*, f_{(-i)j}^*) - \sum_{i\in I} W_i^k(f_i^*, f_{-i}^*) \\
&= \sum_{i\in I} R_i^k(f_{ij}^*, f_{(-i)j}^*, \varphi_{ij}(f_{(-i)j}^*)) - \sum_{i\in I} R_i^k(f_i^*, f_{-i}^*, \varphi_i(f_{-i}^*)) \\
&= \sum_{i\in I} R_i^k(f_{ij}^*, f_{(-i)j}^*, \varphi_{ij}(f_{(-i)j}^*)) - \sum_{i\in I} R_i^k(f_{ij}^*, f_{(-i)j}^*, \varphi_i(f_{-i}^*)) \\
&\quad + \sum_{i\in I} R_i^k(f_{ij}^*, f_{(-i)j}^*, \varphi_i(f_{-i}^*)) - \sum_{i\in I} R_i^k(f_i^*, f_{-i}^*, \varphi_i(f_{-i}^*)) \\
&< \frac{1+\max\{\beta_i\}_{i\in I}}{2}\epsilon_5 + \frac{1-\max_{i\in I}\{\beta_i\}}{6}\epsilon_5 \\
&= \frac{2+\max\{\beta_i\}_{i\in I}}{3}\epsilon_5,
\end{aligned}
$$

which contracts (6). Therefore, for $1 \le k \le K$,

$$
\lim_{j\to\infty} \sum_{i\in I} \varphi_{ij}^k(f_{(-i)j}^*) = \sum_{i\in I} \varphi_i^k(f_{-i}^*).
$$

Because of the condition of aggregate upper semicontinuity,

$$
\limsup_{j\to\infty} \sum_{i\in I} R_i^k(f_{ij}^*, f_{(-i)j}^*, \varphi_i(f_{-i}^*)) \le \sum_{i\in I} R_i^k(f_i^*, f_{-i}^*, \varphi_i(f_{-i}^*)).
$$

By the definition of the mappings $\{R_i^k\}_{i\in I, 1\le k\le K}$,

$$
\begin{aligned}
&\left| \sum_{i\in I} R_i^k(f_{ij}^*, f_{(-i)j}^*, \varphi_{ij}(f_{(-i)j}^*)) - \sum_{i\in I} R_i^k(f_{ij}^*, f_{(-i)j}^*, \varphi_i(f_{-i}^*)) \right| \\
&\le \left\| \varphi_{ij}(f_{(-i)j}^*) - \varphi_i(f_{-i}^*) \right\|_2
\end{aligned}
$$

$$\to 0,$$

where $\|\cdot\|_2$ is the usual Euclidean norm in $\mathbb{R}^K$. Therefore,

$$\limsup_{j\to\infty}\sum_{i\in I} W_{ij}^k(f_{ij}^*, f_{(-i)j}^*) \le \sum_{i\in I} W_i^k(f_i^*, f_{-i}^*).$$

Step 6. In the final step, we verify that $f^* = (f_1^*, \dots, f_n^*)$ is a stationary Markov perfect equilibrium.

First observe that for each $i \in I$,

$$W_i(f_i^*, f_{-i}^*) = R_i(f_i^*, f_{-i}^*, \varphi_i(f_{-i}^*)) = \varphi_i(f_{-i}^*) = \phi_i(f_{-i}^*, \varphi_i(f_{-i}^*)).$$

It follows from that for each $1 \le k \le K$,

$$W_i^k(f_i^*, f_{-i}^*) \le \phi_i^k(f_{-i}^*, \varphi_i(f_{-i}^*)) = \varphi_i^k(f_{-i}^*),$$

and

$$\begin{aligned}\sum_{i\in I} W_i^k(f_i^*, f_{-i}^*) &\ge \limsup_{j\to\infty}\sum_{i\in I} W_{ij}^k(f_{ij}^*, f_{(-i)j}^*)\\ &\ge \limsup_{j\to\infty}\sum_{i\in I}\left[\varphi_{ij}^k(f_{(-i)j}^*) - \frac{1}{j}\right]\\ &= \sum_{i\in I}\varphi_i^k(f_{-i}^*).\end{aligned}$$

Let $(v_1^*, \dots, v_n^*)$ be the continuation payoff generated by $f^* = (f_1^*, \dots, f_n^*)$ via the Bellman equation (1), and v_i^{k*} be the continuation payoff of player i when the common state is ω_k. For $i \in I$ and $1 \le k \le K$, let $\tilde{v}_i^{k*} = \int_{S_i} v_i^{k*}(s_i) P_i(\mathrm{d}s_i|\omega^k)$. Then $\tilde{v}_i^* = (\tilde{v}_i^{1*}, \dots, \tilde{v}_i^{K*}) \in \tilde{V}_i$, and

$$\begin{aligned}\tilde{v}_i^{k*} &= \int_{S_i} v_i^{k*}(s_i) P_i(\mathrm{d}s_i|\omega^k)\\ &= \int_{S_i}\int_{S_{-i}}\int_X U_i(\omega^k, \mathbf{s}, \mathbf{x}, v_i^*) \otimes_{m\in I} f_m^*(\mathrm{d}x_m|\omega^k, s_m) P_{-i}(\mathrm{d}s_{-i}|\omega^k) P_i(\mathrm{d}s_i|\omega^k)\\ &= \int_{S_i}\int_{S_{-i}}\int_X U_i(\omega^k, \mathbf{s}, \mathbf{x}, \tilde{v}_i^*) \otimes_{m\in I} f_m^*(\mathrm{d}x_m|\omega^k, s_m) P_{-i}(\mathrm{d}s_{-i}|\omega^k) P_i(\mathrm{d}s_i|\omega^k)\end{aligned}$$

$$= R_i^k(f_i^*, f_{-i}^*, \tilde{v}_i^*).$$

Thus, $\tilde{v}_i^*$ and $\varphi_i(f_{-i}^*)$ are both fixed points of $R_i(f_i^*, f_{-i}^*, \cdot)$ in $\tilde{V}_i$. Since $R_i(f_i^*, f_{-i}^*, \cdot)$ is a contraction mapping for each $i \in I$, $\tilde{v}_i^* = \varphi_i(f_{-i}^*)$. Therefore, $f^* = (f_1^*, \ldots, f_n^*)$ is a stationary Markov perfect equilibrium.

6.2 Proof of Proposition 1

We shall show that this stochastic dynamic oligopoly model can be viewed as a continuation payoff secure stochastic game satisfying the aggregate upper semicontinuity condition.

We first verify the continuation payoff security. Fix $\epsilon > 0$, firm $i \in I$ and $f_i \in \mathcal{L}_i$. For $j \geq 1$, define a strategy g_i^j of firm i as follows: if $f_i(\omega, s_i) = -1$, then $g_i^j(\omega, s_i) = -1$; if $f_i(\omega, s_i) = (a_{i1}, \ldots, a_{iL}) \neq -1$, then for $1 \leq l \leq L$, $g_{il}^j(\omega, s_i)$, the l-th coordinate of $g_i^j(\omega, s_i)$ (*i.e.*, the price of product l), is

$$\begin{cases} \min\{a_{il} + \frac{1}{j}, \overline{a}_l\} & \text{if } a_{il} \leq c_l(\omega), \\ \max\{a_{il} - \frac{1}{j}, c_l(\omega), \underline{a}_l\} & \text{if } a_{il} > c_l(\omega). \end{cases}$$

It is clear that $|f_{il}(\omega, s_i) - g_{il}^j(\omega, s_i)| \leq \frac{1}{j}$. Since the demand functions $\{D_l(\omega, \cdot)\}_{1 \leq l \leq L, \omega \in \Omega}$, the mappings $\{\xi_{i'l}(\omega, \cdot)\}_{i' \in I, 1 \leq l \leq L, \omega \in \Omega}$, and the state transitions $\{Q(\omega, \cdot)\}_{\omega \in \Omega}$ are continuous in $\mathbf{x}$ and X is a compact set, they are also uniformly continuous. Fix some sufficiently large integer $J \geq \frac{1}{\epsilon}$ such that for any $\mathbf{x}, \mathbf{y}$ with $\|\mathbf{x} - \mathbf{y}\|_2 < \frac{1}{J}$,[20] $|D_l(\omega, \mathbf{x}) - D_l(\omega, \mathbf{y})| < \epsilon$, $|\xi_{i'l}(\omega, \mathbf{x}) - \xi_{i'l}(\omega, \mathbf{y})| < \epsilon$, $\|Q(\omega, \mathbf{x}) - Q(\omega, \mathbf{y})\|_2 < \epsilon$ for each $i' \in I$, $1 \leq l \leq L$, and $\omega \in \Omega$. Hereafter, we focus on some $j > J$.

If $f_i(\omega, s_i) = -1$, then $g_i^j(\omega, s_i) = -1$. For any x_{-i},

$$u_i(\omega, \mathbf{s}, g_i^j(\omega, s_i), x_{-i}) = u_i(\omega, \mathbf{s}, f_i(\omega, s_i), x_{-i}) = s_i$$

[20] If $x_{i'}$ or $y_{i'}$ is -1 for some $i' \in I$, then we view it as $(-1, -1, \ldots, -1) \in \mathbb{R}^L$ without loss.

for an incumbent firm, and

$$u_i(\omega, \mathbf{s}, g_i^j(\omega, s_i), x_{-i}) = u_i(\omega, \mathbf{s}, f_i(\omega, s_i), x_{-i}) = 0$$

for a potential entrant.

Suppose that $f_i(\omega, s_i) = (a_{i1}, \ldots, a_{iL}) \neq -1$. Given the other firms' actions x_{-i}, we consider the profit of a firm from each submarket l (*i.e.*, the submarket for product l). There are four possible cases.

a. Suppose that $f_{il}(\omega, s_i) = a_{il} \leq c_l(\omega)$, and firm i is either a firm with the price strictly higher than the lowest price/market price in the submarket l, or one of the firms with the lowest price in the submarket l. Then $g_{il}^j(\omega, s_i) > a_{il}$. By proposing the price $g_{il}^j(\omega, s_i)$, firm i cannot get any share in this submarket, as there must be some other firm with a price lower than $g_{il}^j(\omega, s_i)$. In addition, even if the opponent firms play some other action y_{-i}, as long as y_{-i} is within some sufficiently small neighborhood $O_{x_{-i}}$ of x_{-i}, firm i still cannot get any share in this submarket.[21] Since $f_{il}(\omega, s_i) \leq c_l(\omega)$, firm i is (weakly) better off by deviating to $g_{il}^j(\omega, s_i)$ in the submarket l.

b. Next, we still consider the possibility that $f_{il}(\omega, s_i) = a_{il} \leq c_l(\omega)$, while firm i is the only firm proposing the lowest price a_{il} in the submarket l. Then firm i gets the nonpositive profit $a_{il} - c_l(\omega)$ with the whole market share $D_l(\omega, \mathbf{x})$ in the submarket l. If firm i deviates to the price $g_{il}^j(\omega, s_i)$, then firm i gets a higher profit $g_{il}^j(\omega, s_i) - c_l(\omega)$ with a smaller (including zero) market share. This observation is still true when the opponent firms play some nearby action y_{-i} from a sufficiently small neighborhood $O_{x_{-i}}$ of x_{-i}. Given other firms' actions $y_{-i} \in O_{x_{-i}}$ and firm i deviating to $g_{il}^j(\omega, s_i)$, (i) firm i is better off when it is still the only firm getting the whole market share, or it has no market share at all (*i.e.*, $g_{il}^j(\omega, s_i)$ is strictly above the market price); (ii) when firm i is one of the firms with the market price $g_{il}^j(\omega, s_i)$, its total profit could decrease in the submarket l given the change of $D_l(\omega, f_i(\omega, s_i), x_{-i})$ to $D_l(\omega, g_i^j(\omega, s_i), y_{-i})$, while the loss is bounded by ϵ due to the choice of J and

[21] In the four possible cases here, we choose the neighborhood $O_{x_{-i}}$ of x_{-i} such that $\|(f_i(\omega, s_i), x_{-i}) - (g_i^j(\omega, s_i), y_{-i})\| < \frac{1}{J}$ for any $y_{-i} \in O_{x_{-i}}$.

$O_{x_{-i}}$.

c. Now we turn to the case that $f_{il}(\omega, s_i) = a_{il} > c_l(\omega)$, and firm i gets a strictly positive market share. By deviating to $g_{il}^j(\omega, s_i) < a_{il}$, firm i is the firm getting the whole market share. This claim still holds when other firms play some action y_{-i} within a sufficiently small neighborhood $O_{x_{-i}}$ of x_{-i}. By the choice of J and $O_{x_{-i}}$, $\|(f_i(\omega, s_i), x_{-i}) - (g_i^j(\omega, s_i), y_{-i})\| < \frac{1}{J}$, which means that $|D_l(\omega, f_i(\omega, s_i), x_{-i}) - D_l(\omega, g_i^j(\omega, s_i), y_{-i})| < \epsilon$. Thus, firm i's total profit from the submarket l either increases, or decreases by no more than ϵ.

d. The last case to consider is that $f_{il}(\omega, s_i) = a_{il} > c_l(\omega)$, and firm i's price in the submarket l, $f_{il}(\omega, s_i)$, is strictly higher than the market price. In this submarket, firm i gets no share and its profit is zero. Since $g_{il}^j(\omega, s_i) \geq c_l(\omega)$, firm i's profit is always nonnegative no matter what actions the other firms choose. In particular, this is true when other firms can choose a nearby action y_{-i} within a sufficiently small neighborhood $O_{x_{-i}}$ of x_{-i}.

Recall the payoff U_i in Section 2.2, which is the sum of the stage profit $u_i(\omega, \mathbf{s}, \mathbf{x})$, and the expected future payoff $\beta_i \sum_{\Omega} \int_{S_i} v_i(\omega', s_i') P_i(\mathrm{d}s_i'|\omega') Q(\omega'|\omega, \mathbf{x})$. By the analysis on the stage profit above, the assumption that $Q(\omega, \cdot)$ is continuous in $\mathbf{x}$, and the construction that $|f_{il}(\omega, s_i) - g_{il}^j(\omega, s_i)| \leq \frac{1}{j}$ for any $i \in I$, $1 \leq l \leq L$, and $(\omega, s_i) \in \Omega \times S_i$, the dynamic oligopoly model is a continuation payoff secure stochastic game.

Next, we verify the condition of aggregate upper semicontinuity. Since the state transition Q is continuous in $\mathbf{x}$, we only need to show that $\sum_{i \in I} u_i(\omega, \mathbf{s}, \mathbf{x})$ is upper semicontinuous in $\mathbf{x}$. It suffices to verify that the total profits of all firms in each submarket is an upper semicontinuous mapping in terms of the action profile.

Given the market state ω and the action profile $\mathbf{x} = (x_1, \ldots, x_n)$ in the current stage, suppose that $\tilde{I} \subseteq I$ is the set of firms with the action -1, and $\overline{I} \subseteq I$ is the set of firms which propose the lowest price α in the submarket l. That is, $x_i = -1$ for each $i \in \tilde{I}$, $a_{il} = \alpha$ for each $i \in \overline{I}$, and $a_{il} > \alpha$ for any $i \in I \setminus (\tilde{I} \cup \overline{I})$. The total profit in the submarket l is $(\alpha - c_l(\omega)) D_l(\omega, \mathbf{x})$. Let $\{\mathbf{x}^j\}$ be a sequence of action profiles converging to $\mathbf{x} = (x_1, \ldots, x_n)$, where $\mathbf{x}^j = (x_1^j, \ldots, x_n^j)$ for $j \geq 1$. For some sufficiently large j, $x_i^j = -1$ for each $i \in \tilde{I}$, and $a_{i'l}^j > a_{il}^j$ for each $i' \in I \setminus (\tilde{I} \cup \overline{I})$ and $i \in \overline{I}$. The total profit in the submarket l is $(\min_{i \in \overline{I}} \{a_{il}^j\} - c_l(\omega)) D_l(\omega, \mathbf{x}^j)$, which clearly converges to $(\alpha - c_l(\omega)) D_l(\omega, \mathbf{x})$. Thus, the aggregate upper semicontinuity condition is satisfied.

To summarize, the stochastic dynamic oligopoly model is a continuation payoff secure stochastic game satisfying the aggregate upper semicontinuity condition. The existence result follows from Theorem 1.

References

Charalambos D. Aliprantis and Kim C. Border, *Infinite Dimensional Analysis: A Hitchhiker's Guide*, Springer, Berlin, 2006.

Rabah Amir, Stochastic games in economics and related fields: An overview, in *Stochastic Games and Applications* (A. Neyman and S. Sorin ed.), NATO Adv. Sci. I. C-MAT, Vol. 570, Kluwer Academic, Dordrecht, 2003, 455–470.

David Blackwell, Discounted dynamic programming, *The Annals of Mathematical Statistics* **36** (1965), 226–235.

Oriol Carbonell-Nicolau and Richard P. McLean, On the existence of Nash equilibrium in Bayesian games, *Mathematics of Operations Research* **43** (2018), 100–129.

Oriol Carbonell-Nicolau and Richard P. McLean, Nash and Bayes–Nash equilibria in strategic-form games with intransitivities, *Economic Theory* **68** (2019), 935–965.

Partha Dasgupta and Eric Maskin, The existence of equilibrium in discontinuous economic games, I: Theory, *The Review of economic studies* **53** (1986), 1–26.

Ulrich Doraszelski and Mark Satterthwaite, Computable Markov-perfect industry dynamics, *The RAND Journal of Economics* **41** (2010), 215–243.

Darrell Duffie, John Geanakoplos, Andreu Mas-Colell, and Andrew McLennan, Stationary Markov equilibria, *Econometrica* **62** (1994), 745–781.

John Duggan, Noisy stochastic games, *Econometrica* **80** (2012), 2017–2045.

Richard Ericson and Ariel Pakes, Markov-perfect industry dynamics: A framework for empirical work, *The Review of economic studies* **62** (1995), 53–82.

Arlington M. Fink, Equilibrium in a stochastic n-person game, *Journal of Science of the Hiroshima University* **28** (1964), 89–93.

Haifeng Fu, Mixed-strategy equilibria and strong purification for games with private and public information, *Economic Theory* **37** (2008), 521–532.

Wei He and Yeneng Sun, Stationary Markov perfect equilibria in discounted stochastic games, *Journal of Economic Theory* **169** (2017), 35–61.

Wei He and Nicholas C. Yannelis, Existence of equilibria in discontinuous Bayesian games, *Journal of Economic Theory* **162** (2016), 181–194.

Anna Jaśkiewicz and Andrzej S. Nowak, Zero-sum stochastic games, in *Handbook of Dynamic Game Theory* (2018a), 215–279.

Anna Jaśkiewicz and Andrzej S. Nowak, Non-zero-sum stochastic games, in *Handbook of Dynamic Game Theory* (2018b), 281–344.

M. Ali Khan, Kali P. Rath and Yeneng Sun, On a private information game without pure strategy equilibria, *Journal of Mathematical Economics* **31** (1999), 341–359.

Yehuda Levy, A discounted stochastic game with no stationary Nash equilibrium: Two examples, *Econometrica* **81** (2013), 1973–2007.

Yehuda Levy and Andrew McLennan, Corrigendum to: "A discounted stochastic game with no stationary Nash equilibrium: Two examples", *Econometrica* **83** (2015), 1237–1252.

Eric Maskin and Jean Tirole, A theory of dynamic oligopoly, I: Overview and quantity competition with large fixed costs, *Econometrica* **56** (1988), 549–569.

Eric Maskin and Jean Tirole, A theory of dynamic oligopoly, II: Price competition, kinked demand curves, and Edgeworth cycles, *Econometrica* **56** (1988), 571–599.

Eric Maskin and Jean Tirole, Markov perfect equilibrium: I. Observable actions, *Journal of Economic Theory* **100** (2001), 191–219.

Ernest Michael, A note on paracompact spaces, *Proceedings of the American Mathematical Society* **4** (1953), 831–838.

Paul R. Milgrom and Robert J. Weber, Distributional strategies for games with incomplete information, *Mathematics of Operations Research* **10** (1985), 619–632.

Paulo K. Monteiro and Frank H. Page Jr, Uniform payoff security and Nash equilibrium in compact games, *Journal of Economic Theory* **134** (2007), 566–575.

Andrzej S. Nowak and T. E. S. Raghavan, Existence of stationary correlated equilibria with symmetric information for discounted stochastic games, *Mathematics of Operations Research* **17** (1992), 519–527.

Pavlo Prokopovych, On equilibrium existence in payoff secure games, *Economic Theory* **48** (2011), 5–16.

Pavlo Prokopovych and Nicholas C. Yannelis, On the existence of mixed strategy Nash equilibria, *Journal of mathematical economics* **52** (2014), 87–97.

Pavlo Prokopovych and Nicholas C. Yannelis, On monotone approximate and exact equilibria of an asymmetric first-price auction with affiliated private information, *Journal of Economic Theory* **184** (2019), 104925.

Philip J. Reny, On the existence of pure and mixed strategy Nash equilibria in discontinuous games, *Econometrica* **67** (1999), 1029–1056.

Philip J. Reny, Nash equilibrium in discontinuous games, *Economic Theory* **61**

(2016), 553–569.

Philip J. Reny, Nash equilibrium in discontinuous games, *Annual Review of Economics*, forthcoming, 2020.

Lloyd Shapley, Stochastic games, *Proceedings of the National Academy of Sciences USA* **39** (1953), 1095–1100.

Yeneng Sun, Integration of correspondences on Loeb spaces, *Transactions of the American Mathematical Society* **349** (1997), 129–153.

Masayuki Takahashi, Equilibrium points of stochastic non-cooperative n-person games, *Journal of Science of the Hiroshima University* **28** (1964), 95–99.

Xavier Vives, Complementarities and games: New developments, *Journal of Economic Literature* **43** (2005), 437–479.