Dynamic Project Assignment

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Abstract

We consider a project assignment problem where a Principal needs to assign multiple projects to a long-lived Agent. The Agent is privately informed about her cost, which evolves stochastically over time. To fully characterize the optimal mechanism, we identify the key trade-off of the Principal, which is the comparison between the benefit from an immediate assignment at high cost and the payoff from a delayed assignment at low cost. We show that the capacity constraint can reduce the payoff of the Principal and create a "hold-up" problem by comparing the limit of the optimal payoffs of the Principal when the capacity constraint is present, with the Principal's optimal payoff when the capacity constraint is absent.

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Keywords: Dynamic Mechanism Design; Capacity Constraint; Hold-up Problem; Deadline

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Contents

1 Introduction

A Principal (he) relies on an Agent (she) to complete multiple potential projects. For example, a firm hires an expert (headhunter) to fill several positions; a teamleader considers assigning a few tasks to a team member; or a retailer needs to outsource some business to a service provider. The Agent has private information on the cost of completing a project: the headhunter is more informed of the difficulty level of finding candidates in the labor market; the team member knows whether she is feeling good or tired; and the service provider has private information about its cost of production. The Principal's problem is that he is uncertain when and how to assign those projects to the Agent is the best course of action.

This paper studies the optimal mechanism for a Principal who has multiple ($m \geq 1$) projects, and needs to hire an Agent to help with those projects. The Agent can finish one project in each stage. The Agent's cost could be high (c_H) or low (c_L), which is persistent and evolves stochastically over time. Hiring the Agent to complete a project yields the Principal a payoff, which is always higher than the Agent's cost. The Principal could pay a certain amount to induce the Agent to take a project in any stage before all the projects are completed.[1] However, taking the Agent's information into account and designing the mechanism strategically would result in payoff gains.

The key messages of our results are that, very generally, (1) the intertemporal trade-off is the main driving force behind the design of the optimal mechanism, and (2) this trade-off can induce a hold-up problem for the Principal, which reduces his payoff and delays the assignments of the projects in the limit.

Intuitively, when the Principal considers whether to assign a project, he would compare the gain from assigning it in the current stage, with the payoff from keeping the project and assigning it in the future. That is, there is an opportunity cost for assigning a project immediately. This observation explains the structure of the optimal mechanism in our model. If the Agent is of low cost, then assigning a project is efficient, and it is optimal for the Principal to follow the efficient assignment rule. On the other hand, when the Agent is of high cost, the trade-off then really matters. To illustrate the idea, we discuss the simple case with only one project in the next two paragraphs. The Principal's problem in this case is essentially an optimal stopping problem.

If the cost c_H is sufficiently high, then it is efficient to keep the project and assign it in the subsequent stages whenever the Agent is of low cost. A simple

[1] For example, a simple strategy is to pay the high cost (c_H), which covers the Agent's cost of taking a project. This strategy always gives the Principal a positive payoff.

mechanism for the Principal is to assign the project and pay c_L to the Agent if she reports low cost, and keep the project otherwise. Obviously, this mechanism is incentive compatible for the Agent and the Principal extracts all the surplus. Since such an assignment rule is efficient, it means that the Principal's payoff is also maximized. Thus, the optimal contract will be fully efficient, and the project will be kept forever along the worst reporting path (the path with only high costs).

The more subtle case is that it is efficient to assign the project even with a high cost. Certainly the Principal could make his decision based on the prior belief without soliciting reports, but this may not be the optimal mechanism. If the Principal simply pays the Agent the reported cost, then it is not incentive compatible for the Agent to adopt the truthtelling strategy. In the optimal mechanism, the Principal needs to delay the assignment of the project given the reporting of high costs for sufficiently long, and the length of delay is chosen so that the Principal's payoff is maximized and misreporting does not benefit the low-cost Agent. As a result, the contract will exhibit inefficiency as the assignment of the project along the reporting path with only high costs will be distorted at later contracting stages.

It is shown that the intuition from the one-project case can be extended to the general setting with multiple projects. That is, the main trade-off is to compare the gain from assigning a project upon the high cost, with the payoff from keeping the project and assigning it in the future based on the low cost. If the latter is higher, then the optimal contract will be fully efficient with the projects being assigned only upon low costs. If the former is higher, then inefficiency will appear in the optimal mechanism in the sense that the assignments of all the projects along the worst reporting path are distorted at later contracting stages. In particular, we show that the Principal shall fix a decreasing sequence of stages $\{T_k^*\}$, where T_k^* is the assignment deadline of the k-th to last project along the worst reporting path. That is, if the Agent always reports the high cost, then the Principal will keep the k-th to last project until the deadline T_k^* is reached.

An important feature of this optimal mechanism is that the deadlines are immune to the number of initial projects. Postponing the assignment of a project not only delays the assignment in the current stage, but also delays the assignments of all the subsequent projects. When there are k projects, intuitively, if the Principal decides to ignore the current project and handle the next $k-1$ projects first, then the earliest time he can come back to assign this project is k-stages later. As a result, the trade-off he would consider is to compare the gain from assigning the project immediately with the gain from keeping the project and assigning it k-stages later. However, this consideration only takes into account the number of available projects, but has nothing to do with the assignment and reporting histories in the previous stages. That is, when determining the deadline for a particular project, the Principal is looking forward rather than looking back.

Finally, we consider the limit setting where the number of initial projects converges to infinity. The model with countably many projects provides a convenient idealization for the model with a large but finite number of projects. The long-term contracting problem without capacity constraint in our setting is similar to that considered in Battaglini (2005). One may hope that the model with constraint would converge to the idealized model when the number of projects increases. Interestingly, this is not true. We show that there is some dissonance between the idealized model and its asymptotic versions. In particular, the payoff of the Principal is reduced and the assignments of the projects along the worst path are further delayed in the limit model with capacity constraint. Loosely, the Principal prefers waiting longer when the constraint is present. Another significant difference is that the projects get eventually assigned in the model without capacity constraint, even along the path with only high costs. This is due to the fact that the only role of the strategical delay in that model is to induce the Agent to report the truth when she is of low cost, and the consideration of the opportunity cost is absent in this setting. However, when the capacity constraint is present, if the cost is very high, then the Principal never assigns any project along the worst path.

The rest of the paper is organized as follows. The remainder of this section provides a review of the related literature. Section 2 introduces the model. Section 3 considers the one-project case in the complete information and incomplete information settings, and identifies the key condition of the paper. Section 4 characterizes the optimal mechanism in the general environment, and compares the Principal's payoffs in the models with and without capacity constraint. Section 5 concludes. The Appendix collects the proofs.

Related Literature.

This paper joins the line of research in dynamic mechanism design, which considers a Principal-agent problem with the Agent's preference evolving stochastically over time; see, for example, Baron and Besanko (1984), Besanko (1985), Courty and Li (2000), Battaglini (2005), Esõ and Szentes (2007) and Pavan, Segal and Toikka (2014). The literature of dynamic mechanism design is too vast to be discussed in the context of this paper, we refer the readers to the recent surveys Pavan (2017) and Bergemann and Välimäki (2018) and the textbook Gershkov and Moldovanu (2014) for more details.[2]

[2]The theory of optimal dynamic mechanism design is significantly developed in the last decade in various environments: *e.g.*, Board (2007) extended Courty and Li (2000) to the case where the sales date

A closely related work to this paper is Battaglini (2005). He considers a nonlinear pricing model in which the buyer's valuation evolves over time according to a commonly known first-order Markov process with two states. As explained before, the no-constraint version of our model can be viewed as a similar problem studied in that paper, which is the dynamic contracting setting with infinitely many projects. Much of the work following Battaglini's paper was focused on the case where the allocation problem is assumed to be time-invariant in the sense that the set of feasible choices in stage t does not depend on the allocation decisions in previous stages. This assumption is often violated in the dynamic environment with capacity constraint. The key difference in our set-up is that we introduce additional constraint on the Principal side. The optimal mechanism therefore may or may not end in finite stages in our setting, while the interaction between the Principal and the Agent continues for infinite stages in Battaglini (2005). An important difficulty, new in this paper, is to characterize the key condition which determines whether or not the Principal should assign projects along the worst path, and to identify the deadline for each project.

Another related strand considers the dynamic allocation problem with fixed capacity of indivisible goods to sell by a (possibly infinite) deadline. This line of research typically considers the setting where potential buyers arrive randomly and the population of privately informed buyers changes over time. Gershkov and Moldovanu (2009) studied a revenue-maximizing monopolist selling several heterogeneous objects to short-lived agents who arrive sequentially, where the arrival of an agent is public information. When the arrivals are private information, Board and Skrzypacz (2016) considered the sales problem of finitely many identical units, and showed that the optimal selling mechanism is a deterministic sequence of posted prices. Gershkov, Moldovanu and Strack (2017) extended the model to cover the case where the buyer's arrival process is initially unknown. Each buyer is assumed to have unit demand and the buyer's privately known valuation does not change over time in those papers. In the current paper, we assume that the Agent can take multiple projects and has stochastically evolving costs.[3]

is an endogenous choice; Board (2008) and Garrett (2016) showed that the optimal price path fluctuates in durable good markets; Armstrong and Zhou (2015) considered a search market and showed how a seller may deter buyers from searching for a better product; and Li and Shi (2017) studied how a seller can disclose additional information to the buyer about her valuation without observing the realization. From a methodological standpoint, the analysis largely relies on the first-order approach. Pavan, Segal and Toikka (2014) provided a general treatment of this approach in the dynamic environment, and obtain necessary results for incentive compatibility.

[3]Garrett (2017) considered a sales problem of non-durable goods where the buyer arrives privately and randomly with evolving private valuations. This paper does not consider the issue of capacity constraint.

2 Model

We consider a discrete-time environment in which a Principal has $m \geq 1$ projects, and needs to hire an Agent to help with those projects. The Agent is able to finish one project in each stage at some cost. In stage t, the cost of the Agent is of one of the two possible types: $c_t \in \{c_H, c_L\}$ ($c_H > c_L > 0$). In the initial stage, the Principal's prior belief is $P(c_1 = c_H) = \lambda_H \in (0, 1)$, and $\lambda_L = 1 - \lambda_H$.

We assume that the distribution of the Agent's cost in stage $t+1$ depends only on her cost in stage t, which means that the Agent's cost c_t in stage-t is a sufficient statistic for her later costs. Denote

$$P[c_{t+1} = c_H | c_t = c_H] = \alpha_H \quad \text{and} \quad P[c_{t+1} = c_H | c_t = c_L] = \alpha_L.$$

Costs are assumed to be persistent in the sense that $0 < \alpha_L \leq \alpha_H < 1$.[4] We adopt the following notations: a sequence of costs from stage 1 to stage t is denoted by $c^t = (c_1, \ldots, c_t)$, and $\{c^0\} = \emptyset$. Throughout this paper, we assume that the cost is the Agent's private information.

In every stage, the Principal can assign a project to the Agent. The payoff of the Principal for a completed project is v ($v > c_H > c_L$). To incentivize the Agent to truthfully report her private cost and finish the assigned project, the Principal can choose a monetary transfer to the Agent. Both the Principal and the Agent are impatient and have a discount factor $\delta \in (0, 1)$.

We use s_t to track the number of projects left unfinished in stage t. That is, $s_1 = m$ indicating the fact that there are m projects in stage 1. If there are m' ($m' \leq m$) projects left in stage t, then $s_t = m'$. If the Principal does not assign a project to the Agent in stage t, then the number of unfinished projects remains the same in the next stage, and hence $s_{t+1} = s_t$. Otherwise, $s_{t+1} = s_t - 1$, which means that one project has been taken by the Agent. Similarly, we use the notation $s^t = (s_1, \ldots, s_t)$ for $t \geq 1$.

Mechanism.

In the first stage, the Principal offers a contract to the Agent. We assume that the Principal can fully commit to a long-term contract. By the revelation principle, we focus on the incentive compatible direct mechanisms without loss of generality. A direct mechanism $\Gamma = (\mathbf{q}, \mathbf{p})$ is a collection of assignment rules $\mathbf{q} = \{q_t\}_{t \geq 1}$ and payment rules $\mathbf{p} = \{p_t\}_{t \geq 1}$. The timing is as follows:

1. In stage 1, a private cost $c_1 \in \{c_H, c_L\}$ is randomly drawn for the Agent. The

[4] That is, the cost process satisfies the first-order stochastic dominance: the probability of high cost tomorrow conditional on today's cost being high is higher than the probability of high cost tomorrow conditional on today's cost being low.

Agent is then asked to make a report $\tilde{c}_1 \in \{c_H, c_L\}$. Based on the report, the Principal assigns a project to the Agent with probability $q_1(\tilde{c}_1; s_1) \in [0,1]$ and chooses a monetary transfer $p_1(\tilde{c}_1; s_1) \in \mathbb{R}$, where $s_1 = m$.

2. In stage $t \geq 1$, if a project is completed, then the tracking number of unfinished projects is deducted by 1 and hence $s_{t+1} = s_t - 1$; otherwise, the number of unfinished projects remains unchanged, and hence $s_{t+1} = s_t$.

3. If $s_t \geq 1$ in stage $t > 1$, then the Agent draws a private cost $c_t \in \{c_H, c_L\}$ following the law of motion $P[c_t|c_{t-1}]$, and is further asked to make a report $\tilde{c}_t \in \{c_H, c_L\}$. Based on the history $\tilde{c}^t = (\tilde{c}^{t-1}, \tilde{c}_t)$ and $s^t = (s^{t-1}, s_t)$, the probability of assigning a project in that stage is $q_t(\tilde{c}^t; s^t) \in [0,1]$, and a transfer $p_t(\tilde{c}^t; s^t) \in \mathbb{R}$ is made.

4. If $s_t = 0$ for some $t > 1$, then $q_t(\tilde{c}^t; s^t) \equiv 0$ and $p_t(\tilde{c}^t; s^t) \equiv 0$ for any $\tilde{c}^t$. This means that all the projects have been completed and the contract is terminated automatically.

In the direct mechanism above, the assignment rule and the payment rule depend on both the reporting history and the history of the tracking numbers of projects. In words, the Principal cares not only about the cost of the Agent, but also about the number of the unfinished projects.

The Agent's problem.

In stage t, given the assignment probability q_t and the transfer p_t, the stage payoff of the Agent is

$$-q_t \cdot c_t + p_t.$$

Fix a mechanism $\Gamma = (\mathbf{q}, \mathbf{p})$. In stage $t \geq 1$, given the history c^{t-1} and s^t, and the cost c_t in the current stage, the Agent's expected payoff hereafter is

$$U(c^{t-1}, c_t; s^t) = \sum_{i \geq 0} \delta^i \mathbb{E}\big[- q_{t+i}(c^{t+i}; s^{t+i})c_t + p_{t+i}(c^{t+i}; s^{t+i})|c_t\big].$$

Given s^t, let $\tilde{s}^{t+1} = (s^t, s_t - 1)$ be a vector of the tracking numbers, where $\tilde{s}_{t+1} = s_t - 1$ means that one project has been assigned in the end of stage t. On the other hand, $\hat{s}^{t+1} = (s^t, s_t)$ represents the history that there is no assignment in stage t and $\hat{s}_{t+1} = s_t$.

For any $t \geq 1$, the Agent is called *incentive compatible* in stage t (IC_t) if for any c^{t-1}, c_t, $\tilde{c}_t$, and s^t,

$$\begin{aligned} &- q_t(c^t; s^t)c_t + p_t(c^t; s^t) + \delta q_t(c^t; s^t)\mathbb{E}\left[U(c^t, c_{t+1}; \tilde{s}^{t+1})|c_t\right] \\ &+ \delta(1 - q_t(c^t; s^t))\mathbb{E}\left[U(c^t, c_{t+1}; \hat{s}^{t+1})|c_t\right] \end{aligned}$$

$$\geq -q_t(c^{t-1},\tilde{c}_t;s^t)c_t + p_t(c^{t-1},\tilde{c}_t;s^t) + \delta q_t(c^{t-1},\tilde{c}_t;s^t)\mathbb{E}\left[U(c^{t-1},\tilde{c}_t,c_{t+1};\tilde{s}^{t+1})|c_t\right]$$
$$+ \delta(1-q_t(c^{t-1},\tilde{c}_t;s^t))\mathbb{E}\left[U(c^{t-1},\tilde{c}_t,c_{t+1};\hat{s}^{t+1})|c_t\right].$$

The left hand side of the inequality gives the payoff of the Agent if she truthfully reports her cost c_t in stage t. It consists of three parts. The first part $-q_t(c^t;s^t)c_t + p_t(c^t;s^t)$ is the stage payoff of the Agent. The second part $\delta q_t(c^t;s^t)\mathbb{E}\left[U(c^t,c_{t+1};\tilde{s}^{t+1})|c_t\right]$ is the discounted future payoff if one project is assigned (with probability $q_t(c^t;s^t)$), while the third term $\delta(1-q_t(c^t;s^t))\mathbb{E}\left[U(c^t,c_{t+1};\hat{s}^{t+1})|c_t\right]$ is the discounted future payoff if no project is assigned (with probability $1-q_t(c^t;s^t)$). The Agent's overall payoff is the summation of the three parts. The right hand side of the inequality gives the payoff of the Agent if she misreports $\tilde{c}_t$. The Agent is said to be incentive compatible if it is optimal for her to choose truthtelling in any stage.

For $t \geq 1$, the Agent is said to be *individually rational* in stage t (IR_t) if for any c^t and s^t,

$$U(c^t;s^t) \geq 0.$$[5]

That is, the Agent has the choice to walk away from the contract at any time if the expected continuation payoff falls below the reservation value 0.

The Principal's problem.

Given that the Agent truthfully reports her private costs, the expected payoff of the Principal is

$$\mathbb{E}\left[\sum_{t\geq 1}\delta^{t-1}(vq_t(c^t;s^t) - p_t(c^t;s^t))\right].$$

The Principal's aim is to maximize his expected payoff subject to all (IC_t) and (IR_t) for $t \geq 1$.

3 Illustration: One Project

We shall first work with the setting that the Principal has only one project at hand ($m = 1$), and illustrate several key points of the main result via this simple case.

Benchmark: Complete information

We start with the benchmark case that the Principal has complete information; that is, the Principal observes the cost of the Agent. This assumption simplifies the analysis by ignoring the Agent's incentive problem.[6] The question is then reduced

[5] Notice that the IC_t and IR_t constraints are satisfied automatically if $s_t = 0$.

[6] The Principal still needs to incentivize the Agent to participate in the mechanism.

to be an individual decision problem of the Principal: When shall the Principal assign the project to the Agent? Since the Principal knows the cost of the Agent, the Principal could simply pay the Agent her cost whenever the Agent is asked to work on the project. As a result, the Principal's aim is to maximize the surplus.

In any stage, the cost of the Agent is either high or low. If the Agent is of low cost, it must be of the Principal's interest to assign the project to the Agent. The trade-off would arise if the Agent has high cost. In this case, the Principal needs to compare the surplus from two possible choices:

1. If the project is assigned immediately, then the surplus is $v - c_H$.
2. If the Principal chooses not to assign the project in the current stage, then it would not be optimal for him to assign the project at c_H in any later stage either (because of the stationary structure). That is, the Principal shall assign the project to the Agent in the earliest subsequent stage in which the cost is low. The surplus is hence

$$\sum_{t\geq 0} \delta^{t+1} P(c_H|c_H)^t P(c_L|c_H)(v - c_L) = \frac{\delta(1-\alpha_H)}{1-\delta\alpha_H}(v - c_L).$$

We have

$$v - c_H \leq \frac{\delta(1-\alpha_H)}{1-\delta\alpha_H}(v - c_L)$$

if and only if

$$v - c_H \leq \delta\Big((v - c_L)(1-\alpha_H) + (v - c_H)\alpha_H\Big).$$

The following assumption summarizes the above comparison, which is the key condition of this paper.

Assumption 1.

a. $v - c_H \leq \delta\Big((v - c_L)(1-\alpha_H) + (v - c_H)\alpha_H\Big).$

b. $v - c_H > \delta\Big((v - c_L)(1-\alpha_H) + (v - c_H)\alpha_H\Big).$

The next proposition is then straightforward.

Proposition 1. *The Principal's optimal (efficient) mechanism under complete information is characterized by the following assignment rule.*

1. For any c^{t-1} *and* s^t *such that* $s_t = 1$*, let*

$$q_t(c^{t-1}, c_L; s^t) = 1.$$

2. *Under Assumption 1 (a), for any s^t such that $s_t = 1$,*

$$q_t(c_H^t; s^t) = 0,$$

where c_H^t is a vector with t components of c_H. That is, the Principal never assigns the project along the path $(c_H, c_H, \ldots)$.

3. *Under Assumption 1 (b),*

$$q_1(c_H; s^1) = 1.$$

That is, the Principal assigns the project in the initial stage.

The above proposition shows that in the efficient assignment rule, the project is assigned given a low cost. Given the path with high costs, however, the assignment could happen either immediately or never, depending on whether the future expected surplus based on the low cost is higher than $v - c_H$, the surplus in the current stage if the project is assigned immediately.

Incomplete information

Hereafter, we consider the case that the Principal cannot observe the private cost of the Agent. When there is only one project, the decision for the Principal would be simple if the Agent is competent (low cost c_L): the Principal shall assign the project to the Agent immediately (as in the complete information setting). However, when the Agent is of high cost c_H, the Principal faces an intertemporal trade-off: he can either assign the project to the Agent in that stage, or keep the project and wait for the Agent's cost to become low in the future. While this comparison is familiar from the complete information setting, the analysis is more complicated and the optimal mechanism could be very different.

In the optimal mechanism, we show that the Principal shall set a deadline T_1^{1*}. In any stage before the deadline T_1^{1*}, the Principal assigns the project to the Agent only if the Agent reports low cost in that stage. If the Agent keeps reporting that her cost is "c_H", then the Principal shall wait until the deadline, and assign the project to the Agent in stage T_1^{1*} regardless of the Agent's report. This deadline T_1^{1*} might be finite or infinite, depending on the comparison between the current surplus and future surplus (*i.e.*, Assumption 1). The proofs of Lemma 1 and Proposition 2 are left in Section 6.1.

The following result is useful by showing how the Agent's expected continuation payoff depends on her cost in the current stage.

Lemma 1. *Fix an incentive compatible and individually rational mechanism Γ with the assignment rule $\mathbf{q}$. Given the history c^{t-1} and s^t in stage $t \geq 1$, we have*

$$U(c^{t-1}, c_L; s^t) - U(c^{t-1}, c_H; s^t) \tag{1}$$

$$\geq (c_H - c_L) \sum_{i=0}^{\infty} \delta^i (\alpha_H - \alpha_L)^i q_{t+i}(c^{t-1}, c_H^{i+1}; s^{t+i}) \prod_{1 \leq j \leq i} [1 - q_{t+j-1}(c^{t-1}, c_H^j; s^{t+j-1})].$$

Remark 1. *The inequality above gives a lower bound of the additional payoff that the Agent could enjoy when the cost is low rather than high. In stage t, in order for the Agent not to misreport high cost when she has low cost, the payoff of the Agent from truthtelling must be no less than the payoff from misreporting. This would result in the following inequality:*

$$\begin{aligned} & U(c^{t-1}, c_L; s^t) - U(c^{t-1}, c_H; s^t) \\ \geq & (c_H - c_L) q_t(c^{t-1}, c_H; s^t) \\ & + \delta(1 - q_t(c^{t-1}, c_H; s^t))(\alpha_H - \alpha_L)[U(c^{t-1}, c_H, c_L; \hat{s}^{t+1}) - U(c^{t-1}, c_H, c_H; \hat{s}^{t+1})], \end{aligned}$$

where $\hat{s}^{t+1} = (s^t, s_t)$. The right hand side of the inequality is a convex combination of two terms. The first term is the additional surplus $c_H - c_L$ for having low cost rather than high cost, if the project is assigned in the current stage. The second term is the difference of the expected continuation payoffs between having low cost and high cost in stage t, multiplied by the probability that the assignment of the project is postponed to the future. Applying the same inequality to the difference $U(c^{t-1}, c_H, c_L; \hat{s}^{t+1}) - U(c^{t-1}, c_H, c_H; \hat{s}^{t+1})$ repeatedly, we get the right hand side of Inequality (1).

As standard in the literature, we show that in the optimal mechanism, the expected continuation payoff of the Agent is 0 at the worse cost c_H. In addition, Inequality (1) is binding. The term $\delta^i(\alpha_H - \alpha_L)^i(c_H - c_L)$ can be understood as the expected rent of having low cost rather than high cost in stage t, if the project is assigned in stage $t + i$. The summation in Inequality (1) is a convex combination of those potential additional payoffs, with the parameter $q_{t+i}(c^{t-1}, c_H^{i+1}; s^{t+i}) \prod_{1 \leq j \leq i}[1 - q_{t+j-1}(c^{t-1}, c_H^j; s^{t+j-1})]$ being the probability that this assignment happens exactly i stages later. Thus, the lower bound is simply the aggregate of the expected payoffs from each subsequent stage.

Note that $U(c_H; s^1) \geq 0$ by the participation constraint. Given Inequality (1), the expected payoff of the Agent is

$$\begin{aligned} & \lambda_L U(c_L; s^1) + \lambda_H U(c_H; s^1) = \lambda_L (U(c_L; s^1) - U(c_H; s^1)) + U(c_H; s^1) \\ \geq & \lambda_L (c_H - c_L) \sum_{i=0}^{\infty} \delta^i (\alpha_H - \alpha_L)^i q_{i+1}(c_H^{i+1}; s^{i+1}) \prod_{1 \leq j \leq i} [1 - q_j(c_H^j; s^j)]. \end{aligned}$$

The expected surplus is

$$\sum_{c_1\in\{c_H,c_L\}} \lambda(c_1)\Big\{q_1(c_1;s_1)(v-c_1)$$
$$+\delta[1-q_1(c_1;s_1)]\big(W(c_1,c_H;s_1,s_1)P(c_H|c_1)+W(c_1,c_L;s_1,s_1)P(c_L|c_1)\big)\Big\},$$

where $W(c^2;s^2)$ is the expected surplus in stage 2 given $(c^2;s^2)$. The payoff of the Principal must be greater than the expected surplus less the lower bound of the Agent's expected rents.

Lemma 2. *Given an incentive compatible and individually rational mechanism* Γ *with the assignment rule* $\mathbf{q}$*, the expected payoff of the Principal is no greater than*

$$E_P^1 = \sum_{c_1\in\{c_H,c_L\}} \lambda(c_1)\Big\{q_1(c_1;s_1)(v-c_1)$$
$$+\delta[1-q_1(c_1;s_1)]\big(W(c_1,c_H;s_1,s_1)P(c_H|c_1)+W(c_1,c_L;s_1,s_1)P(c_L|c_1)\big)\Big\}$$
$$-\lambda_L(c_H-c_L)\sum_{i=0}^{\infty}\delta^i(\alpha_H-\alpha_L)^i q_{i+1}(c_H^{i+1};s^{i+1})\prod_{1\le j\le i}[1-q_j(c_H^j;s^j)].$$

Note that the expected surplus in the second stage is argumented by the probability $1-q_1(c_1;s_1)$. That is, if $q_1(c_1;s_1)=1$, then the mechanism will end in the first stage and the subsequent costs are irreverent, since there is only one project. If $q_1(c_1;s_1)<1$, then the subsequent costs are still relevant, and hence the expected continuation surplus needs to be adjusted accordingly.

Next, we characterize the optimal mechanism for the one-project case in the proposition below.

Proposition 2. *The optimal mechanism is characterized by the following assignment rule. In addition, the Principal's payoff is given by* E_P^1 *in Lemma 2.*

1. *For any* c^{t-1} *and* s^t *such that* $s_t=1$*, let*

$$q_t(c^{t-1},c_L;s^t)=1.$$

2. *Under Assumption 1 (a), for any* s^t *such that* $s_t=1$*,*

$$q_t(c_H^t;s^t)=0.$$

That is, the Principal never assigns the project along the path $(c_H,c_H,\ldots)$*. Let* $T_1^{1*}=\infty$*.*

3. Suppose that Assumption 1 (b) holds. Denote T_1^{1*} as the smallest t such that

$$\begin{aligned} v - c_H &\geq \delta\Big((v - c_L)(1 - \alpha_H) + (v - c_H)\alpha_H\Big) \\ &+ \frac{\lambda_L}{\lambda_H}(c_H - c_L)\Big[1 - \delta(\alpha_H - \alpha_L)\Big]\left(\frac{\alpha_H - \alpha_L}{\alpha_H}\right)^{t-1}. \end{aligned} \tag{2}$$

Let

$$q_t(c_H^t; s^t) = \begin{cases} 0, & t < T_1^{1*}, \\ 1, & t = T_1^{1*}. \end{cases}$$

Remark 2. *The above proposition characterizes the optimal mechanism via the assignment rule. In the optimal mechanism, Inequality (1) is binding and the maximal payoff of the Principal is E_P^1 from Lemma 2.*

The optimal assignment rules in claims (1-2) of Proposition 2 are the same as that in Proposition 1, which are efficient. It is not hard to understand this similarity. Intuitively, the Agent does not have any incentive to misreport low cost when she has high cost. Thus, the Principal does not need to leave any rent to the high-cost Agent since she must be truthtelling when reporting c_L, which implies that the assignment rule should be surplus maximizing if the report is c_L. If Assumption 1 (a) holds, then it is efficient to hold the project given a high cost. In this case, the Agent is incentive compatible and is left zero rents at both costs. Thus, it is optimal for the Principal to adopt this assignment rule since he gets the total surplus. That is, the Principal's interest is aligned with the goal of surplus maximization.

Suppose that Assumption 1 (b) holds. Then the Principal cannot achieve efficiency and incentive compatibility at the same time, and extract the full surplus. Otherwise, the Agent may find it beneficial to report c_H when having c_L. The Principal uses a delayed (inefficient) assignment rule along the path $(c_H, c_H, \ldots)$ such that the low-cost Agent does not benefit from misreporting. The Agent is left positive rents given a low cost.

The proof proceeds by first identifying an assignment rule which maximizes the upper bound E_P^1. We then construct the corresponding payments such that the mechanism with the given assignment and payment rules respect all the incentive constraints and participation constraints. As a result, this mechanism gives the maximal payoff that the Principal could achieve.

The upper bound E_P^1 is the expected surplus less the lower bound of the Agent's expected rents. Note that only the assignment along the path $(c_H, c_H, \ldots)$ is relevant for calculating this lower bound. Whenever a low cost c_L is reported, the Principal only needs to maximize the surplus. Thus, Claim (1) in Proposition 2 is straightforward: as the surplus $v - c_L$ (if the project is assigned immediately) is always higher than the discounted future surplus, the Principal shall assign the project immediately whenever a low cost c_L is drawn.

For Proposition 2 (2-3), we need to compare $v-c_H$ and $\delta\big((v-c_L)(1-\alpha_H)+(v-c_H)\alpha_H\big)$. If the former is less than the latter, it means that the discounted future surplus based on the low cost is more valuable than the surplus in this stage given the current high cost. In this case, the Principal finds it more attractive to delay at high cost, and chooses never to assign the project along the path $(c_H, c_H, \ldots)$; this gives Proposition 2 (2). If $v - c_H$ is higher than $\delta\big((v-c_L)(1-\alpha_H)+(v-c_H)\alpha_H\big)$, then Proposition 2 (3) characterizes the (finite) optimal stopping stage T_1^{1*}.[7]

One way to understand the assignment rule in Proposition 2 (2-3) is as follows. Along the path $(c_H, c_H, \ldots)$, the Principal needs to decide at which stage the project should be assigned (or never). This is an optimal stopping problem for the Principal. Suppose that the Principal chooses some $t' \geq 1$, and lets $q_t(c_H^t; s^t) = 0$ for any $t < t'$ and $q_{t'}(c_H^{t'}; s^{t'}) = 1$. Then the relevant part of E_P^1 for the Principal from this choice is

$$\begin{aligned}
&\lambda_H \delta \sum_{k\geq 0}^{t'-2} [\delta^k P(c_H|c_H)^k] P(c_L|c_H)(v-c_L) + \lambda_H \delta^{t'-1} P(c_H|c_H)^{t'-1}(v-c_H) \\
&- \lambda_L (c_H - c_L)\delta^{t'-1}\left(P(c_H|c_H) - P(c_H|c_L)\right)^{t'-1} \\
=\ & \lambda_H \delta(1-\alpha_H)(v-c_L)\frac{1-\delta^{t'-1}\alpha_H^{t'-1}}{1-\delta\alpha_H} + \lambda_H \delta^{t'-1}\alpha_H^{t'-1}(v-c_H) \\
&- \lambda_L (c_H - c_L)\delta^{t'-1}\left(\alpha_H - \alpha_L\right)^{t'-1}. \qquad (3)
\end{aligned}$$

The first term describes the expected surplus in the case that the initial cost is c_H and it changes to c_L in no later than the stage t'. The second term is the part of the expected surplus for the case that the cost in the first t' stages are all "c_H". The last term is the rent left to the Agent. If one chooses $t' = \infty$, then the payoff in (3) is reduced to be

$$\lambda_H \delta(1-\alpha_H)(v-c_L)\frac{1}{1-\delta\alpha_H}. \qquad (4)$$

It is easy to check that (3) $\leq$ (4) for any t' if and only if Assumption 1 (a) holds. As the result, the Principal will choose never to assign the project if the discounted future surplus at low cost is always more attractive given the current cost c_H, which gives (2) in Proposition 2. If Assumption 1 (b) holds, we then show that the payoff in (3) is single peaked in terms of the stopping stage t'. Thus, a (finite)

[7] In particular, as $t \to \infty$, the right-hand side of Inequality (2) approaches $\delta\big((v-c_L)(1-\alpha_H)+(v-c_H)\alpha_H\big)$, which implies that Inequality (2) must be satisfied for sufficiently large finite t.

optimal stopping stage exists, which is characterized in Proposition 2 (3).

In summary, identifying the assignment choice along the path $(c_H, c_H, \ldots)$ in Proposition 2 (2-3) involves two comparisons. The Principal needs to guarantee that given an assignment in some (finite or infinite) stage t, (a) the difference between the surplus and the Agent's rent is always nonnegative; (b) the chosen stage to assign the project needs to be optimal for the Principal among all the possible stages. While the first part is familiar from Battaglini (2005), the second part is new to the present paper.

Remark 3. *Consider the special case without serial correlation: $\alpha_H = \alpha_L$. Inequality (2) is satisfied for any $t \geq 1$, which implies that the project assignment happens immediately in stage 1 in the case of Proposition 2 (3): $q_1(c_H; s_1) = 1$. Intuitively, no serial correlation means that the cost in stage t cannot convey any information for the cost in stage $t+1$. That means the value of the current cost does not have any further effect on future surplus. In the case with serial correlation, the assignment of the project may be delayed even if the surplus is more valuable than the discounted future surplus, since the misreporting of the low-cost Agent not only affects the assignment (and hence the surplus) in the current stage, but also distorts the belief of the Principal on future surplus. Thus, the Principal needs to use a delayed deadline as a threat to deter the Agent from lying. In the case without serial correlation, the Principal always has the correct belief on future costs, and the project is assigned immediately.*

4 Main Results

In this section, we shall analyze the Principal's problem subject to the general constraint ($m \geq 1$).

The optimal mechanism is characterized by the assignment rule in Proposition 3. It is shown that when the cost is low, the Principal shall follow the efficient assignment rule and assign a project to the Agent in that stage. When the Agent's cost is high, however, inefficiency may arise in the optimal mechanism. The key trade-off for the Principal is to compare the payoff from assigning the project in the current stage with the payoff from reassigning it in the future; in other words, the Principal needs to take the opportunity cost into consideration. If delaying the assignment is efficient given a high cost, then the Principal shall adopt this efficient assignment rule and be able to extract full surplus. To the contrary, when it is inefficient to delay upon a high cost, the Principal stills needs to strategically delay the assignment to induce the Agent to follow the truthtelling strategy. Thus, the optimal mechanism will exhibit inefficiency and the Agent is left with positive rents. The proofs of Lemma 3 and Proposition 3 are left in the Appendix.

Furthermore, the important role of the capacity constraint is highlighted in Proposition 4 via the comparison between the results in the settings with capacity constraint (m is finite) and without capacity constraint (*i.e.*, $m = +\infty$). It is shown that the Principal's optimal payoff/mechanism in the former case may not converge to those in the latter case. The Principal has a hold-up problem in the sense that the limit of the assignments of the projects along the worst path $(c_H, c_H, \ldots)$ is in general strictly delayed compared with that in the setting without capacity constraint. In addition, the limit of the Principal's payoffs in the former case is less than that in the latter case. This discontinuity at infinity is mainly driven by the intertemporal trade-off when the number of projects is limited.

4.1 The General Setting

The following lemma generalizes Lemmas 1 and 2 to the general environment. It provides a lower bound for the additional payoff of the Agent when the cost is low rather than high, and also an upper bound for the Principal's payoff.

Lemma 3. *Fix an incentive compatible and individually rational mechanism* Γ *with the assignment rule* $\mathbf{q}^*$.

1. *Given the history* c^{t-1} *and* s^t *with* $s_t = m \geq 1$ *in stage* $t \geq 1$*, we have*

$$
\begin{aligned}
& U(c^{t-1}, c_L; s^t) - U(c^{t-1}, c_H; s^t) \qquad (5) \\
\geq\ & (c_H - c_L) \sum_{i_1=0}^{\infty} \Big\{ \delta^{i_1} (\alpha_H - \alpha_L)^{i_1} \, q^*_{t+i_1}(c^{t-1}, c_H^{i_1+1}; s^{t+i_1}) \cdot \\
& \prod_{1 \leq j \leq i_1} [1 - q^*_{t+j-1}(c^{t-1}, c_H^{j}; s^{t+j-1})] \Big\} \\
& + (c_H - c_L) \sum_{\substack{i_1 \geq 0 \\ i_2 \geq 1}} \Big\{ \delta^{i_1+i_2} (\alpha_H - \alpha_L)^{i_1+i_2} \, q^*_{t+i_1}(c^{t-1}, c_H^{i_1+1}; s^{t+i_1}) \\
& \quad \cdot q^*_{t+i_1+i_2}(c^{t-1}, c_H^{i_1+i_2+1}; s^{t+i_1+i_2}) \prod_{\substack{1 \leq j \leq i_1+i_2 \\ j \neq i_1+1}} [1 - q^*_{t+j-1}(c^{t-1}, c_H^{j}; s^{t+j-1})] \Big\} \\
& + \cdots \\
& + (c_H - c_L) \sum_{\substack{i_1 \geq 0 \\ i_2 \geq 1 \\ \cdots \\ i_m \geq 1}} \Big\{ \delta^{\sum_{k=1}^m i_k} (\alpha_H - \alpha_L)^{\sum_{k=1}^m i_k} \prod_{k=1}^{m} q^*_{t+\sum_{l=1}^k i_l}(c^{t-1}, c_H^{\sum_{l=1}^k i_l + 1}; s^{t+\sum_{l=1}^k i_l})
\end{aligned}
$$

$$\cdot \prod_{\substack{1\le j\le \sum_{k=1}^{m} i_k \\ j\neq i_1+1 \\ j\neq i_1+i_2+1 \\ \cdots \\ j\neq \sum_{k=1}^{m-1} i_k+1}} [1-q^*_{t+j-1}(c^{t-1},c^j_H;s^{t+j-1})]\Bigg\}.$$

Let $E^m_A(c^{t-1},s^t)$ *be the summation on the right hand side of the above inequality.*

2. *The expected payoff of the Principal is no greater than*

$$\begin{aligned} E^m_P = \sum_{c_1\in\{c_H,c_L\}} & \lambda(c_1)\Big\{ q^*_1(c_1;s_1)(v-c_1) \\ & + \delta q^*_1(c_1;s_1)\left(W(c_1,c_H;s_1,s_1-1)P(c_H|c_1) + W(c_1,c_L;s_1,s_1-1)P(c_L|c_1)\right) \\ & + \delta[1-q^*_1(c_1;s_1)]\big(W(c_1,c_H;s_1,s_1)P(c_H|c_1) + W(c_1,c_L;s_1,s_1)P(c_L|c_1)\big)\Big\} \\ & - \lambda_L\cdot E^m_A(c^0,s^1). \end{aligned}$$

Remark 4. *Though the term* $E^m_A(c^{t-1},s^t)$ *in Lemma 3 (1) seems complicated at the first glance, it has a very nice structure for interpretation. It is shown in the proof of Proposition 3 below that in the optimal mechanism, the Agent gets zero payoff at high cost and* $E^m_A(c^{t-1},s^t)$ *at low cost. In addition, Inequality (5) is binding, and the payoff of the Principal is* E^m_P. *For* $1\le k\le m$, *the* k*-th summation in* $E^m_A(c^{t-1},s^t)$ *corresponds to the expected rents pinned down to the path* $(c^{t-1},c_H,c_H,\ldots)$ *and the* k*-th project. In particular, every such summation in* $E^m_A(c^{t-1},s^t)$ *represents the marginal payoff of the Agent due to the corresponding project. In the* k*-th summation, we use the notations* $i_1,i_2,\ldots,i_k$ *to describe that the first project is assigned* i_1 *stages later, the second project is assigned* i_1+i_2 *stages later, so on and so forth. Then each term* $\delta^{\sum_{k'=1}^{k} i_{k'}}(\alpha_H-\alpha_L)^{\sum_{k'=1}^{k} i_{k'}}(c_H-c_L)$ *in the* k*-th summation is the expected rents of having low cost rather than high cost in stage* t, *if the* k*-th project is assigned exactly* $\sum_{k'=1}^{k} i_{k'}$ *stages later. The* k*-th summation is hence simply the aggregate of the expected rents from every such stage.*

The proof of Lemma 3 (1) involves two levels of induction. In the first level, the argument is based on the induction on m, *the number of projects. Lemma 1 shows that the claim is true when there is one project. We shall assume that the claim is true in the case with* m' *projects and then show that it is still true with* $m'+1$ *projects. To complete the argument in the case with* $m'+1$ *projects, we need another induction argument to deal with the incentive of the low-cost Agent and simplify the expression of the lower bound. The argument of the second-level induction is similar to that in the proof of Lemma 1.*

Before stating the main result, it is useful to extend the discounted future surplus to the general setting. Let

$$f_k(c_1) = \delta^{k-1} \sum_{c_2,\ldots,c_k \in \{c_H, c_L\}} \left[(v - c_k) \cdot \prod_{1 \le j \le k-1} P(c_{j+1}|c_j) \right].$$

Note that $f_1(c_1) = v - c_1$. The term $f_k(c_1)$ is the discounted expected surplus $k-1$ stages later when the initial cost is c_1 for $k \ge 1$.

The following proposition characterizes the optimal mechanism in the general setting.

Proposition 3. *The optimal mechanism is characterized by the following assignment rule.*

1. *For any c^{t-1} and s^t such that $s_t \ge 1$, let*

$$q_t^*(c^{t-1}, c_L; s^t) = 1.$$

2. *Under Assumption 1 (a), for any c^{t-1} and s^t such that $s_t \ge 1$,*

$$q_t^*(c^{t-1}, c_H; s^t) = 0.$$

3. *Suppose that Assumption 1 (b) holds.*

   (a) *For any $c^{t-1} \ne c_H^{t-1}$ and s^t such that $s_t \ge 1$,*

$$q_t^*(c^{t-1}, c_H; s^t) = 1.$$

   (b) *For $1 \le k \le m$, denote T_k^{m*} as the smallest t such that*

$$\begin{aligned} v - c_H \quad & \ge \delta\Big(f_k(c_L)(1-\alpha_H) + (v - c_H)\alpha_H \Big) \\ & + \frac{\lambda_L}{\lambda_H}(c_H - c_L)\Big[1 - \delta(\alpha_H - \alpha_L)\Big]\left(\frac{\alpha_H - \alpha_L}{\alpha_H}\right)^{t-1}. \end{aligned} \tag{6}$$

   Then for $c^t = c_H^t$ and s^t such that $s_t = k \ge 1$, let

$$q_t^*(c_H^t; s^t) = \begin{cases} 0, & t < T_k^{m*}; \\ 1, & t \ge T_k^{m*}. \end{cases}$$

   That is, the Principal shall set a deadline T_k^{m*} for the k-th to last project, and wait until stage T_k^{m*} to assign that project along the path $(c_H, c_H, \ldots)$.

Before providing more discussions in remarks below, we first present a corollary which summarizes two important properties of the deadlines in Proposition 3 (3b). First, the deadlines are independent of the number of original projects. Second, the deadlines follow the reverse order.

Corollary 1. *Suppose that Assumption 1 (b) holds. Consider Proposition 3 (3b).*

1. *For $1 \le k \le m$, the deadline T_k^{m*} does not depend on m.*
2. *For any $c \in \{c_H, c_L\}$, $\{f_k(c)\}$ is a decreasing sequence and converges to 0 as $k \to \infty$. It implies that the sequence $\{T_k^{m*}\}$ is decreasing in terms of k for any m.*

Proof. (1) By Inequality (6), T_k^{m*} does not depend on m for any k.

(2) It is obvious that

$$v - c_L > \delta\big((v - c_L)(1 - \alpha_L) + (v - c_H)\alpha_L\big).$$

By assumption,

$$v - c_H > \delta\Bigg((v - c_L)(1 - \alpha_H) + (v - c_H)\alpha_H\Bigg)$$

Iterating these two inequalities, $\{f_k(c)\}$ is a decreasing sequence. As $k \to \infty$, δ^k converges to 0, which implies that $f_k(c)$ converges to 0 for any $c \in \{c_H, c_L\}$. The sequence $\{T_k^{m*}\}$ is then decreasing since $\{f_k(c_L)\}$ is decreasing in k. □

Remark 5 (Efficiency)**.** *The Principal adopts the efficient assignment rule in Proposition 3 (1-2). The intuitions for these two claims are similar to that for Proposition 2 (1-2). When Assumption 1 (b) holds, however, there could be two possible cases.*

Along the path c^{t-1} which is not c_H^{t-1}, the Principal assigns a project in every stage as long as all the projects have not been completed. One can imagine that after the Agent reveals herself to be of low cost, there is no residual asymmetric information. Before the realization of the cost in the following stage, the Principal can delegate all the projects to the Agent so that the Agent benefits directly from the completion of those projects. The Principal simply charges the Agent his expected payoff before this delegation.

Along the path $c^{t-1} = c_H^{t-1}$, the Principal shall delay the assignment of the projects to deter the Agent from lying, subject to the deadlines determined in Inequality (6). To be more precise, in the latter case, when the Principal has k projects, he shall assign a project immediately along the path c_H^{t-1} if the stage T_k^{m*}

has been reached, and hold the projects otherwise.[8]

Note that if Assumption 1 (b) holds, then the efficient rule is to assign a project in each stage regardless of the cost. As a result, though the assignment rule in (3a) is efficient, inefficiency appears in (3b). However, since the sequence $\{T_k^{m*}\}$ is decreasing in k by Corollary 1, the assignment rule must be efficient after the furthest deadline T_1^{m*} in (3b).[9] *This implies that the optimal mechanism will converge to the efficient one in the long run.*[10]

Remark 6. *To understand Inequality (6), note that the only difference between Inequality (2) and Inequality (6) is that on the right side of latter inequality, $v - c_L$ (i.e., $f_1(c_L)$) is replaced by $f_k(c_L)$. Recall that the key when deriving Proposition 2 (3) is to compare the current surplus $v - c_H$ given a high cost, with the future surplus only based on the low cost. In particular, when the project is not assigned in the current stage, the Principal can reassign the project in the next stage. When there are multiple projects, if the Principal assigns the k-th to last project given a high cost, he would get the payoff from the surplus $v - c_H$ in the current stage and then the surplus from the next $k - 1$ projects. If the Principal decides to keep the k-th to last project and adopt the same assignment rule for the next $k - 1$ projects, then the earliest time he can reassign the holding project is k stages later. As a result, he shall compare the current surplus $v - c_H$ given a high cost, with the expected surplus $f_k(c_L)$ from k stages later at a low cost.*

4.2 Discussion: The Limit Case

In the following, we compare the optimal mechanism in Proposition 3 with the results in the long-term contracting problem without capacity constraint. We summarize the main results in the setting without capacity constraint in the following lemma; see Battaglini (2005) for more details.

Since T_k^{m*} does not depend on m by Corollary 1, we simplify the notation by omitting the superscript m. Hereafter, we write T_k^{m*} as T_k^*.

Lemma 4. *Fix an incentive compatible and individually rational mechanism Γ with the assignment rule $\mathbf{q}^\infty$.*

[8]It is possible that the deadline T_k^{m*} has already been passed when it is the first time that the Principal has k projects. For example, imagine that the Principal has four projects to finish and the deadline T_2^{4*} for the second to last project is 2. Then the earliest possible stage for the Principal to assign the second to last project is the third stage. Since the deadline T_2^{4*} must have been passed when the Principal has the opportunity to assign that project, he just assigns it to the Agent regardless of the report whenever he has a chance to do it.

[9]By definition, the Principal assigns a project in each stage after stage T_1^{m*} regardless of the cost.

[10]This is consistent with the observation in Battaglini (2005). For more discussions, see Section III(A) therein.

1. *The optimal mechanism is characterized by the following assignment rule.*
   (a) *For any c^{t-1}, let*
   $$q_t^\infty(c^{t-1}, c_L) = 1.$$
   (b) i. *For any $c^{t-1} \neq c_H^{t-1}$,*
   $$q_t^\infty(c^{t-1}, c_H) = 1.$$
   ii. *For $1 \leq k \leq m$, denote T_∞ as the smallest t such that*
   $$v - c_H \quad \geq \frac{\lambda_L}{\lambda_H}(c_H - c_L)\left(\frac{\alpha_H - \alpha_L}{\alpha_H}\right)^{t-1}. \tag{7}$$
   Then for $c^t = c_H^t$, let
   $$q_t^\infty(c_H^t) = \begin{cases} 0, & t < T_\infty; \\ 1, & t \geq T_\infty. \end{cases}$$
   That is, T_∞ is the deadline for the first assignment to happen along the path $(c_H, c_H, \ldots)$. If c_L has appeared before or the deadline T_∞ has been reached, then the Principal will assign a project to the Agent in each stage regardless of the cost.
2. *Given the history c^{t-1} in stage $t \geq 1$, we have*
$$\begin{aligned} & U(c^{t-1}, c_L) - U(c^{t-1}, c_H) \\ = & (c_H - c_L)\sum_{i=0}^{\infty} \delta^i (\alpha_H - \alpha_L)^i q_{t+i}^\infty(c^{t-1}, c_H^{i+1}). \end{aligned}$$
   Let E_A^∞ be the summation on the right hand side of the inequality above.
3. *The expected payoff of the Principal is*
$$\begin{aligned} E_P^\infty = & \sum_{c_1 \in \{c_H, c_L\}} \lambda(c_1)\Big\{q_1^\infty(c_1)(v - c_1) \\ & + \delta\left(W(c_1, c_H)P(c_H|c_1) + W(c_1, c_L)P(c_L|c_1)\right)\Big\} - \lambda_L \cdot E_A^\infty. \end{aligned}$$

It is natural to conjecture that the results in the problem with many projects would approximate the results in the problem without constraint. In other words, one may expect that the optimal payoff of the Principal in Lemma 3 and the optimal mechanism in Proposition 3 would converge to those in Lemma 4 when the number of projects increases. The following proposition shows that this asymptotic result may not be true.

Proposition 4. *The sequence $\{E_P^{m*}\}$ is increasing in m. Let $E_P^{\infty *}$ and T_∞^* be the limits of the sequences $\{E_P^{m*}\}$ and $\{T_m^*\}$, respectively. We have $E_P^{\infty *} \le E_P^\infty$ and $T_\infty^* \ge T_\infty$.*

Proof. A Principal with $m+1$ projects can always abandon one project and guarantee himself the payoff E_P^{m*}. As a result, $E_P^{m+1*} \ge E_P^{m*}$, and hence the sequence $\{E_P^{m*}\}$ is increasing in m. By the same logic, $E_P^{m*} \le E_P^\infty$ for any $m \ge 1$. Thus, $\{E_P^{m*}\}$ is convergent. Let $E_P^{\infty *}$ be the limit of the sequence $\{E_P^{m*}\}$. It is clear that $E_P^{\infty *} \le E_P^\infty$.

Since T_∞^* is the limit of the decreasing sequence $\{T_m^*\}$, we then have that T_∞^* is the smallest t such that

$$v - c_H \ge \frac{\lambda_L}{\lambda_H}(c_H - c_L)\frac{\left[1-\delta(\alpha_H-\alpha_L)\right]}{1-\delta\alpha_H}\left(\frac{\alpha_H-\alpha_L}{\alpha_H}\right)^{t-1}. \tag{8}$$

Since T_∞ is the smallest t such that

$$v - c_H \ge \frac{\lambda_L}{\lambda_H}(c_H - c_L)\left(\frac{\alpha_H-\alpha_L}{\alpha_H}\right)^{t-1},$$

$T_\infty^* \ge T_\infty$ as

$$\frac{\alpha_H-\alpha_L}{\alpha_H} < 1 \text{ and } \frac{1-\delta(\alpha_H-\alpha_L)}{1-\delta\alpha_H} \ge 1.$$

□

Remark 7. *In Proposition 4, we show that the equilibrium payoff of the Principal E_P^{m*} and the optimal mechanism in Proposition 3 may not converge to the equilibrium payoff E_P^∞ and the optimal mechanism in the case without capacity constraint.*

If Assumption 1 (a) holds, then the Principal never assigns a project given the high cost when the capacity constraint is present. The Principal, as a result, cannot extract the stage surplus $v - c_H$ from a high-cost Agent even though she has low cost in some previous stage. On the other hand, when the capacity constraint is absent, the Principal will assign a project to the Agent in each stage regardless of the cost as long as the low cost has appeared before, which means that the Principal is able to extract the stage surplus $v - c_H$ from a high-cost Agent.

Suppose that Assumption 1 (b) holds. Note that T_∞ in the case without capacity constraint does not depend on the discount factor δ, while the limit T_∞^* does depend on δ. The right hand side of Inequality (8) is increasing in δ. The deadline T_∞^* is greater than T_∞, and they coincide only when δ is sufficiently small, say $\delta = 0$. This means that the assignments of the projects along the path $(c_H, c_H, \ldots)$ will be in general further delayed in the case with capacity constraint.

A significant difference between the analysis in the settings with and without capacity constraint is whether the Principal needs to take into account the opportunity cost. In the former case, the Principal compares the payoff by assigning one project in the current stage at high cost, with the payoff by holding the project and reassigning it in the future. Such kind of consideration does not exist in the case without constraint.

5 Conclusion

In this paper, we address the question how a Principal assigns multiple projects to an Agent with changing costs in a dynamic environment. We fully characterize the optimal dynamic mechanism. It is shown that the key in the construction of the optimal mechanism is the intertemporal trade-off; that is, the Principal needs to compare the payoff from assigning the project in the current stage with the payoff from reassigning it in the future. This trade-off induces interesting properties on the assignment rule, and shapes the dynamics of the optimal mechanism accordingly. In particular, when the capacity constraint is present, the Principal may face a hold-up problem of the assignments along the worst path $(c_H, c_H, \ldots)$, which in turn reduces the payoff of the Principal. In reality, the Principal often has the capacity constraint for various (budget/headcount/quota) reasons. This paper contributes to the understanding of such environments.

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6 Appendix

6.1 Proofs of Lemma 1 and Proposition 2

We prove Lemma 1 by modifying the argument in Battaglini (2005).

Proof of Lemma 1.

Given the history c^{t-1} and s^t for $t \geq 1$. Suppose that $s_t = 1$. That is, the project has not been completed in stage t. Recall that $\tilde{s}^{t+1} = (s^t, s_t - 1)$ and $\hat{s}^{t+1} = (s^t, s_t)$ for $t \geq 1$. In order for the Agent not to misreport high cost when she has low cost in stage t, we must have

$$\begin{aligned}
&U(c^{t-1}, c_L; s^t) \\
\geq& -q_t(c^{t-1}, c_H; s^t)c_L + p_t(c^{t-1}, c_H; s^t) \\
&+ \delta q_t(c^{t-1}, c_H; s^t)\mathbb{E}\left[U(c^{t-1}, c_H, c_{t+1}; \tilde{s}^{t+1})|c_L\right] \\
&+ \delta(1 - q_t(c^{t-1}, c_H; s^t))\mathbb{E}\left[U(c^{t-1}, c_H, c_{t+1}; \hat{s}^{t+1})|c_L\right] \\
=& U(c^{t-1}, c_H; s^t) + (c_H - c_L)q_t(c^{t-1}, c_H; s^t) \\
&+ \delta q_t(c^{t-1}, c_H; s^t)(\alpha_H - \alpha_L)[U(c^{t-1}, c_H, c_L; \tilde{s}^{t+1}) - U(c^{t-1}, c_H, c_H; \tilde{s}^{t+1})] \\
&+ \delta(1 - q_t(c^{t-1}, c_H; s^t))(\alpha_H - \alpha_L)[U(c^{t-1}, c_H, c_L; \hat{s}^{t+1}) - U(c^{t-1}, c_H, c_H; \hat{s}^{t+1})].
\end{aligned}$$

Since $\tilde{s}^{t+1} = (s^t, s_t - 1)$ and $s_t - 1 = 0$. We have

$$U(c^{t-1}, c_H, c_L; \tilde{s}^{t+1}) = U(c^{t-1}, c_H, c_H; \tilde{s}^{t+1}) = 0,$$

and hence

$$\begin{aligned}
&U(c^{t-1}, c_L; s^t) - U(c^{t-1}, c_H; s^t) \\
\geq& (c_H - c_L)q_t(c^{t-1}, c_H; s^t) \\
&+ \delta(1 - q_t(c^{t-1}, c_H; s^t))(\alpha_H - \alpha_L)[U(c^{t-1}, c_H, c_L; \hat{s}^{t+1}) - U(c^{t-1}, c_H, c_H; \hat{s}^{t+1})].
\end{aligned}$$

The above inequality gives a lower bound of the additional payoff the Agent could enjoy when the cost is low rather than high.

Suppose that for some $i' \geq 0$,

$$\begin{aligned}
&U(c^{t-1}, c_L; s^t) - U(c^{t-1}, c_H; s^t) \\
\geq& (c_H - c_L)\sum_{i=0}^{i'} \delta^i(\alpha_H - \alpha_L)^i q_{t+i}(c^{t-1}, c_H^{i+1}; s^{t+i}) \prod_{1 \leq j \leq i} [1 - q_{t+j-1}(c^{t-1}, c_H^j; s^{t+j-1})] \\
&+ \delta^{i'+1}(\alpha_H - \alpha_L)^{i'+1} \prod_{1 \leq j \leq i'+1} [1 - q_{t+j-1}(c^{t-1}, c_H^j; s^{t+j-1})]\cdot \\
&[U(c^{t-1}, c_H^{i'+1}, c_L; \hat{s}^{t+i'+1}) - U(c^{t-1}, c_H^{i'+1}, c_H; \hat{s}^{t+i'+1})].
\end{aligned}$$

By the induction argument,

$$U(c^{t-1}, c_L; s^t) - U(c^{t-1}, c_H; s^t)$$

$$
\begin{aligned}
&\geq (c_H - c_L)\sum_{i=0}^{i'+1}\delta^i(\alpha_H-\alpha_L)^i q_{t+i}(c^{t-1},c_H^{i+1};s^{t+i})\prod_{1\leq j\leq i}[1-q_{t+j-1}(c^{t-1},c_H^j;s^{t+j-1})]\\
&\quad+\delta^{i'+2}(\alpha_H-\alpha_L)^{i'+2}\prod_{1\leq j\leq i'+2}[1-q_{t+j-1}(c^{t-1},c_H^j;s^{t+j-1})]\cdot\\
&\quad[U(c^{t-1},c_H^{i'+2},c_L;\hat{s}^{t+i'+2})-U(c^{t-1},c_H^{i'+2},c_H;\hat{s}^{t+i'+2})].
\end{aligned}
$$

Taking i' to infinity, since $[U(c^{t-1},c_H^{i'+2},c_L;\hat{s}^{t+i'+2})-U(c^{t-1},c_H^{i'+2},c_H;\hat{s}^{t+i'+2})]$ is bounded, we have

$$
\begin{aligned}
&U(c^{t-1},c_L;s^t)-U(c^{t-1},c_H;s^t)\\
&\geq (c_H-c_L)\sum_{i=0}^{\infty}\delta^i(\alpha_H-\alpha_L)^i q_{t+i}(c^{t-1},c_H^{i+1};s^{t+i})\prod_{1\leq j\leq i}[1-q_{t+j-1}(c^{t-1},c_H^j;s^{t+j-1})].
\end{aligned}
$$

This completes the proof. □

Proof of Proposition 2.

The proof proceeds in two steps. We first show that the assignment rule given in Proposition 2 maximizes E_P^1. Then we construct the payment rule and check that the corresponding mechanism satisfies all incentive constraints and participation constraints.

Step 1. Recall that

$$
\begin{aligned}
E_P^1 = &\sum_{c_1\in\{c_H,c_L\}}\lambda(c_1)\Big\{q_1(c_1;s_1)(v-c_1)\\
&+\delta[1-q_1(c_1;s_1)]\big(W(c_1,c_H;s_1,s_1)P(c_H|c_1)+W(c_1,c_L;s_1,s_1)P(c_L|c_1)\big)\Big\}\\
&-\lambda_L(c_H-c_L)\sum_{i=0}^{\infty}\delta^i(\alpha_H-\alpha_L)^i q_{i+1}(c_H^{i+1};s^{i+1})\prod_{1\leq j\leq i}[1-q_j(c_H^j;s^j)].
\end{aligned}
$$

We shall check that the assignment rule given by claims (1-3) in Proposition 2 maximizes E_P^1.

(1) This case is obvious. For any c^{t-1} and s^t such that $s_t=1$, since

$$v-c_L>\delta(W(c^{t-1},c_L,c_H;s^{t+1})\alpha_L+W(c^{t-1},c_L,c_L;s^{t+1})(1-\alpha_L)),$$

we have that $q_t(c^{t-1},c_L;s^t)=1$.

(2-3) Suppose that the Principal chooses some $t'\geq 1$, and lets $q_t(c_H^t;s^t)=0$ for any $t<t'$ and $q_{t'}(c_H^{t'};s^{t'})=1$. Note that no assignment will happen after stage t' since the Principal assigns the project in stage t' regardless of the report. As a result, we can ignore the surplus in E_P^1 which is from the path with the initial cost

being c_L, and focus on the following part:

$$
\begin{aligned}
&\lambda_H \delta \sum_{k\geq 0}^{t'-2} [\delta^k P(c_H|c_H)^k] P(c_L|c_H)(v-c_L) + \lambda_H \delta^{t'-1} P(c_H|c_H)^{t'-1}(v-c_H) \\
&- \lambda_L (c_H - c_L)\delta^{t'-1} (\alpha_H - \alpha_L)^{t'-1} \\
=& \lambda_H \delta (1-\alpha_H)(v-c_L) \frac{1-\delta^{t'-1}\alpha_H^{t'-1}}{1-\delta\alpha_H} + \lambda_H \delta^{t'-1}\alpha_H^{t'-1}(v-c_H) \\
&- \lambda_L (c_H - c_L)\delta^{t'-1} (\alpha_H - \alpha_L)^{t'-1}. \qquad (9)
\end{aligned}
$$

If the Principal chooses never to assign the project along the path $(c_H, c_H, \ldots)$ (i.e., $q_t^*(c_H^t; s^t) = 0$ for any t), then the relevant part in the Principal's payoff is

$$
\lambda_H \delta (1-\alpha_H)(v-c_L)\frac{1}{1-\delta\alpha_H}. \qquad (10)
$$

The difference of the payoffs for these two choices is

$$
\begin{aligned}
(9) - (10) =& -\lambda_H \delta (1-\alpha_H)(v-c_L)\frac{\delta^{t'-1}\alpha_H^{t'-1}}{1-\delta\alpha_H} + \lambda_H \delta^{t'-1}\alpha_H^{t'-1}(v-c_H) \\
&- \lambda_L (c_H - c_L)\delta^{t'-1} (\alpha_H - \alpha_L)^{t'-1} \\
=& \lambda_H \delta^{t'-1}\alpha_H^{t'-1}\Bigg[(v-c_H) - \frac{\delta(1-\alpha_H)}{1-\delta\alpha_H}(v-c_L) \\
&- \frac{\lambda_L}{\lambda_H}(c_H-c_L)\left(\frac{\alpha_H-\alpha_L}{\alpha_H}\right)^{t'-1}\Bigg],
\end{aligned}
$$

which is negative for any t' if

$$
v - c_H \leq \delta\Big((v-c_L)(1-\alpha_H) + (v-c_H)\alpha_H\Big).
$$

In this case, the Principal should choose never to assign the project along the path $(c_H, c_H, \ldots)$. Thus, for any s^t such that $s_t = 1$, $q_t(c_H^t; s^t) = 0$. This gives the assignment rule in Proposition 2 (2).

If

$$
v - c_H > \delta\Big((v-c_L)(1-\alpha_H) + (v-c_H)\alpha_H\Big),
$$

then

$$
(v-c_H) - \frac{\delta(1-\alpha_H)}{1-\delta\alpha_H}(v-c_L) > \frac{\lambda_L}{\lambda_H}(c_H-c_L)\left(\frac{\alpha_H-\alpha_L}{\alpha_H}\right)^{t'-1}
$$

holds for sufficiently large t'. As a result, the Principal should choose to assign the project in some (finite) stage. The question is then reduced to pick a stage t' such that (9) is maximized.

Denote $a = \frac{\lambda_H}{\delta\alpha_H}[(v-c_H) - \frac{\delta(1-\alpha_H)(v-c_L)}{1-\delta\alpha_H}]$, $b = \delta\alpha_H$, $c = \frac{\lambda_L}{\delta(\alpha_H-\alpha_L)}(c_H-c_L)$, $d = \delta(\alpha_H - \alpha_L)$. Define a function as $g(t) = ab^t - cd^t$. Taking the first order derivative of g with respect to t, one gets $g'(t) = (a\ln b)b^t - (c\ln d)d^t$. Then $g'(t) \geq 0$ if and only if $t \leq \ln(\frac{a \ln b}{c \ln d}) / \ln \frac{d}{b}$. As a result, g is increasing until $\ln(\frac{a \ln b}{c \ln d}) / \ln \frac{d}{b}$ and then decreasing. Notice that (9) − (10) $= ab^{t'} - cd^{t'}$, and the Principal's payoff is the summation of $ab^{t'} - cd^{t'}$ and a constant (10). Thus, the Principal's payoff is single-peaked: increasing in terms of t' until $\ln(\frac{a \ln b}{c \ln d}) / \ln \frac{d}{b}$, and then (strictly) decreasing afterwards. To identify the optimal stage to assign the project along the path $(c_H, c_H, \ldots)$, we need to find the smallest t such that $g(t-1) \geq g(t)$, which implies that

$$
\begin{aligned}
v - c_H \geq \delta\Big(& (v - c_L)(1 - \alpha_H) + (v - c_H)\alpha_H \Big) \\
& + \frac{\lambda_L}{\lambda_H}(c_H - c_L)\Big[1 - \delta(\alpha_H - \alpha_L)\Big]\left(\frac{\alpha_H - \alpha_L}{\alpha_H}\right)^{t-1}.
\end{aligned}
$$

This gives the assignment rule in Proposition 2 (3).

Step 2. Next, we construct the payment rule and check that the corresponding mechanism satisfies all the incentive constraints and participation constraints.

Given c^{t-1} and s^t with $s_t = 1$,

$$
U(c^{t-1}, c_L; s^t) =
$$

$$
(c_H - c_L)\sum_{i=0}^{\infty} \delta^i (\alpha_H - \alpha_L)^i q_{t+i}(c^{t-1}, c_H^{i+1}; s^{t+i}) \prod_{1 \leq j \leq i} [1 - q_{t+j-1}(c^{t-1}, c_H^j; s^{t+j-1})]
$$

and

$$
U(c^{t-1}, c_H; s^t) = 0.
$$

Let

$$
p_t(c^{t-1}, c_t; s^t) =
$$

$$
U(c^{t-1}, c_t; s^t) + q_t(c^{t-1}, c_t; s^t)c_t - \delta(1 - q_t(c^{t-1}, c_t; s^t))\mathbb{E}(U(c^{t-1}, c_t, c_{t+1}; \hat{s}^{t+1})|c_t).
$$

It is immediate to check that all the participation constraints are satisfied and the Agent gets the payoff $U(c^{t-1}, c_t; s^t)$ at the cost c_t given c^{t-1} and s^t. Following the argument in the proof of Lemma 1, the Agent with low cost would not report high cost. We only need to check that the Agent with high cost would not report low cost. Given c^{t-1} and s^t, if the Agent has cost c_H but reports c_L, then her payoff is

$$
\begin{aligned}
& - q_t(c^{t-1}, c_L; s^t)c_H + p_t(c^{t-1}, c_L; s^t) \\
& + \delta(1 - q_t(c^{t-1}, c_L; s^t))\mathbb{E}\left[U(c^{t-1}, c_L, c_{t+1}; \hat{s}^{t+1})|c_H\right] \\
= {} & U(c^{t-1}, c_L; s^t) - (c_H - c_L) \\
\leq {} & 0 \\
= {} & U(c^{t-1}, c_H; s^t).
\end{aligned}
$$

The first equality holds since $q_t(c^{t-1}, c_L; s^t) = 1$. The inequality is true since $U(c^{t-1}, c_L; s^t) \leq c_H - c_L$ by definition. This completes the proof. □

6.2 Proofs of Lemma 3 and Proposition 3

Proof of Lemma 3.

(1) We adopt an induction argument. By Lemma 1, the claim would be true if there is only one project left. Suppose that the claim holds in the case with $m'-1$ projects. We consider the case with m' projects. Given the history c^{t-1} and s^t for $t \geq 1$. Suppose that $s_t = m'$. Recall that $\tilde{s}^{t+1} = (s^t, s_t - 1)$ and $\hat{s}^{t+1} = (s^t, s_t)$. In order for the Agent not to misreport high cost when she has low cost in stage t, we have

$$\begin{aligned}
&U(c^{t-1}, c_L; s^t) \\
\geq & -q_t(c^{t-1}, c_H; s^t)c_L + p_t(c^{t-1}, c_H; s^t) \\
&+ \delta q_t(c^{t-1}, c_H; s^t)\mathbb{E}\left[U(c^{t-1}, c_H, c_{t+1}; \tilde{s}^{t+1})|c_L\right] \\
&+ \delta(1 - q_t(c^{t-1}, c_H; s^t))\mathbb{E}\left[U(c^{t-1}, c_H, c_{t+1}; \hat{s}^{t+1})|c_L\right] \\
= & U(c^{t-1}, c_H; s^t) + (c_H - c_L)q_t(c^{t-1}, c_H; s^t) \\
&+ \delta q_t(c^{t-1}, c_H; s^t)(\alpha_H - \alpha_L)[U(c^{t-1}, c_H, c_L; \tilde{s}^{t+1}) - U(c^{t-1}, c_H, c_H; \tilde{s}^{t+1})] \\
&+ \delta(1 - q_t(c^{t-1}, c_H; s^t))(\alpha_H - \alpha_L)[U(c^{t-1}, c_H, c_L; \hat{s}^{t+1}) - U(c^{t-1}, c_H, c_H; \hat{s}^{t+1})].
\end{aligned}$$

That is,

$$\begin{aligned}
&U(c^{t-1}, c_L; s^t) - U(c^{t-1}, c_H; s^t) \\
\geq & (c_H - c_L)q_t(c^{t-1}, c_H; s^t) \\
&+ \delta q_t(c^{t-1}, c_H; s^t)(\alpha_H - \alpha_L)[U(c^{t-1}, c_H, c_L; \tilde{s}^{t+1}) - U(c^{t-1}, c_H, c_H; \tilde{s}^{t+1})] \\
&+ \delta(1 - q_t(c^{t-1}, c_H; s^t))(\alpha_H - \alpha_L)[U(c^{t-1}, c_H, c_L; \hat{s}^{t+1}) - U(c^{t-1}, c_H, c_H; \hat{s}^{t+1})].
\end{aligned}$$

Suppose that for some $i' \geq 0$,

$$\begin{aligned}
&U(c^{t-1}, c_L; s^t) - U(c^{t-1}, c_H; s^t) \\
\geq & (c_H - c_L)\sum_{i=0}^{i'} \delta^i(\alpha_H - \alpha_L)^i q_{t+i}(c^{t-1}, c_H^{i+1}; s^{t+i}) \prod_{1\leq j\leq i}[1 - q_{t+j-1}(c^{t-1}, c_H^j; s^{t+j-1})] \\
&+ \sum_{i=0}^{i'}\Bigg\{\delta^{i+1}(\alpha_H - \alpha_L)^{i+1} q_{t+i}(c^{t-1}, c_H^{i+1}; s^{t+i}) \prod_{1\leq j\leq i}[1 - q_{t+j-1}(c^{t-1}, c_H^j; s^{t+j-1})] \cdot \\
&[U(c^{t-1}, c_H^{i+1}, c_L; s^{t+i+1}) - U(c^{t-1}, c_H^{i+1}, c_H; s^{t+i+1})]\Bigg\} \\
&+ \delta^{i'+1}(\alpha_H - \alpha_L)^{i'+1} \prod_{1\leq j\leq i'+1}[1 - q_{t+j-1}(c^{t-1}, c_H^j; s^{t+j-1})] \cdot \\
&[U(c^{t-1}, c_H^{i'+1}, c_L; \hat{s}^{t+i'+1}) - U(c^{t-1}, c_H^{i'+1}, c_H; \hat{s}^{t+i'+1})].
\end{aligned}$$

Applying the induction argument to $i'+1$, and then take i' to the positive infinity, we have

$$U(c^{t-1}, c_L; s^t) - U(c^{t-1}, c_H; s^t)$$

$$\geq (c_H - c_L) \sum_{i=0}^{\infty} \delta^i (\alpha_H - \alpha_L)^i q_{t+i}(c^{t-1}, c_H^{i+1}; s^{t+i}) \prod_{1 \leq j \leq i} [1 - q_{t+j-1}(c^{t-1}, c_H^j; s^{t+j-1})]$$

$$+ \sum_{i=0}^{\infty} \Bigg\{ \delta^{i+1} (\alpha_H - \alpha_L)^{i+1} q_{t+i}(c^{t-1}, c_H^{i+1}; s^{t+i}) \prod_{1 \leq j \leq i} [1 - q_{t+j-1}(c^{t-1}, c_H^j; s^{t+j-1})] \cdot$$

$$[U(c^{t-1}, c_H^{i+1}, c_L; s^{t+i+1}) - U(c^{t-1}, c_H^{i+1}, c_H; s^{t+i+1})] \Bigg\}.$$

Recall that we have assumed Inequality (5) to hold in the case with $m' - 1$ projects. In the second summation of the right hand side of the above inequality, one (and only one) project has been assigned with probability $q_{t+i}(c^{t-1}, c_H^{i+1}; s^{t+i})$. Thus, the tracking number s_{t+i+1} in this summation is $m' - 1$. Applying Inequality (5) to $U(c^{t-1}, c_H^{i+1}, c_L; s^{t+i+1}) - U(c^{t-1}, c_H^{i+1}, c_H; s^{t+i+1})$, we then get Inequality (5) for the case with m' projects. This completes the induction argument.

(2) Note that $U(c_H; s^1) \geq 0$ by the participation constraint. Following the same argument as in Section 3, $E_A^m(c^0, s^1)$ is the lower bound of the Agent's payoff when the cost is low. Then $\lambda_L \cdot E_A^m(c^0, s^1)$ is the lower bound of the Agent's expected rents. The payoff of the Principal must be greater than the expected surplus less the lower bound of the Agent's expected rents. □

Proof of Proposition 3.

We shall proceed in two steps. The first step is to find the assignment rules which maximize E_P^m. Then a payment rule is constructed so that the corresponding mechanism satisfies all the incentive constraints and participation constraints. Though the second step is standard, the first step here is more difficult, as the assignment of a project may influence the assignment path of the remaining projects.

Step 1. We shall check that the assignment rule given in Proposition 3 maximizes E_P^m.

(1) Suppose that in stage t there are k projects left, and $c_t = c_L$. No matter whether one project is assigned in this stage or not, the Principal can always adopt the same strategy for the next $k - 1$ projects.

1. If one project is assigned in stage t, then the $k-1$ projects are all the projects left.
2. If no project is assigned in this stage, then the $k-1$ projects are the k-th-to-last project to the second-to-last project.

If the Principal adopts the same strategy in these two situations, then the surplus contributed by these $k - 1$ projects are the same. Since

$$v - c_L > \delta\Big((v - c_L)(1 - \alpha_L) + (v - c_H)\alpha_L\Big),$$

the surplus contributed by an assigned project in stage t (*i.e.*, case (1)) is strictly higher than the surplus contributed by the last project (*i.e.*, case (2)). As a result, the Principal shall assign one project as long as the the cost is low.

(2) Suppose that

$$v - c_H \leq \delta\Big((v - c_L)(1 - \alpha_H) + (v - c_H)\alpha_H\Big).$$

We need to distinguish two cases: (a) $c^{t-1} \neq c_H^{t-1}$; (b) $c^{t-1} = c_H^{t-1}$.

(2.a) We first work with the case that $c^{t-1} \neq c_H^{t-1}$. Consider the case that there is only one project left in stage t. Suppose that the Principal chooses some $t' \geq 1$, and lets $q_{t+\hat{t}-1}(c^{t-1}, c_H^{\hat{t}}; s^{t+\hat{t}-1}) = 0$ for any $1 \leq \hat{t} < t'$ and $q_{t+t'-1}(c^{t-1}, c_H^{t'}; s^{t+t'-1}) = 1$. The part of surplus in E_P^m from this choice would be

$$\begin{aligned}
&\delta \sum_{k \geq 0}^{t'-2} [\delta^k P(c_H|c_H)^k] P(c_L|c_H)(v - c_L) + \delta^{t'-1} P(c_H|c_H)^{t'-1}(v - c_H) \\
&= \delta(1 - \alpha_H)(v - c_L)\frac{1 - \delta^{t'-1}\alpha_H^{t'-1}}{1 - \delta\alpha_H} + \delta^{t'-1}\alpha_H^{t'-1}(v - c_H).
\end{aligned}$$

If the Principal chooses never to assign the project along the path $(c^{t-1}, c_H, c_H, \ldots)$, then the relevant part in E_P^m is

$$\delta(1 - \alpha_H)(v - c_L)\frac{1}{1 - \delta\alpha_H}.$$

If Assumption 1 (a) holds, then the latter is higher than the former for any t', which implies that the Principal should choose never to assign the last project given a report c_H.

Suppose that the Principal decides that he shall never assign the last $r \geq 1$ projects given a report c_H. Let $\phi_r(c)$ be the expected surplus when there are r projects, the current cost is c, and the Principal only assigns the project at c_L. Then

$$\phi_r(c_H) = \delta P(c_L|c_H)\phi_r(c_L) + \delta P(c_H|c_H)\phi_r(c_H),$$

which implies that

$$\phi_r(c_H) = \frac{\delta(1 - \alpha_H)}{1 - \delta\alpha_H}\phi_r(c_L).$$

Consider the case that there are $r + 1$ projects left in stage t. Suppose that the Principal chooses some $t' \geq 1$, and lets $q_{t+\hat{t}-1}(c^{t-1}, c_H^{\hat{t}}; s^{t+\hat{t}-1}) = 0$ for any $1 \leq \hat{t} < t'$ and $q_{t+t'-1}(c^{t-1}, c_H^{t'}; s^{t+t'-1}) = 1$, when there are $r + 1$ projects. The part of surplus in E_P^m from this choice is

$$\begin{aligned}
&\delta \sum_{k \geq 0}^{t'-2} [\delta^k P(c_H|c_H)^k] P(c_L|c_H)[(v - c_L) + \delta(1 - \alpha_L)\phi_r(c_L) + \delta\alpha_L\phi_r(c_H)] \\
&+ \delta^{t'-1} P(c_H|c_H)^{t'-1}[(v - c_H) + \delta(1 - \alpha_H)\phi_r(c_L) + \delta\alpha_H\phi_r(c_H)].
\end{aligned}$$

If the Principal chooses never to assign the project along the path $(c^{t-1}, c_H, c_H, \ldots)$, then the relevant part in E_P^m is

$$\delta(1-\alpha_H)\frac{1}{1-\delta\alpha_H}[(v-c_L)+\delta(1-\alpha_L)\phi_r(c_L)+\delta\alpha_L\phi_r(c_H)].$$

By substituting $\phi_r(c_H) = \frac{\delta(1-\alpha_H)}{1-\delta\alpha_H}\phi_r(c_L)$, it is straightforward to check if Assumption 1 (a) holds, then the latter is higher than the former for any t', which implies that the Principal should choose never to assign the $r+1$-th to last project given a report c_H. This completes the induction argument.

(2.b) Suppose that $c^{t-1} = c_H^{t-1}$. The proof of this part follows a similar induction argument as that in (2.a). The argument for the one-project case is the same as that in the proof of Proposition 2 (2). Suppose that the Principal decides that he shall never assign the last $r \geq 1$ projects given a report c_H. Again, consider the case that there are $r+1$ projects left in stage t. Suppose that the Principal chooses some $t' \geq 1$, and lets $q_{t+\hat{t}-1}(c_H^{t-1}, c_H^{\hat{t}}; s^{t+\hat{t}-1}) = 0$ for any $1 \leq \hat{t} < t'$ and $q_{t+t'-1}(c_H^{t-1}, c_H^{t'}; s^{t+t'-1}) = 1$, when there are $r+1$ projects. The surplus in E_P^m from this choice would be

$$\begin{aligned}
&\delta\lambda_H\sum_{k\geq 0}^{t'-2}[\delta^k P(c_H|c_H)^k]P(c_L|c_H)[(v-c_L)+\delta(1-\alpha_L)\phi_r(c_L)+\delta\alpha_L\phi_r(c_H)]\\
&+\lambda_H\delta^{t'-1}P(c_H|c_H)^{t'-1}[(v-c_H)+\delta(1-\alpha_H)\phi_r(c_L)+\delta\alpha_H\phi_r(c_H)]\\
&-\lambda_L(c_H-c_L)\delta^{t+t'-1}(\alpha_H-\alpha_L)^{t+t'-1}.
\end{aligned}$$

If the Principal chooses never to assign the project along the path $(c_H^{t-1}, c_H, c_H, \ldots)$, then the relevant part in E_P^m is

$$\delta(1-\alpha_H)\frac{1}{1-\delta\alpha_H}[(v-c_L)+\delta(1-\alpha_L)\phi_r(c_L)+\delta\alpha_L\phi_r(c_H)].$$

Once again, if Assumption 1 (a) holds, then the latter is higher than the former for any t', which implies that the Principal should choose never to assign the $r+1$-th to last project given a report c_H. This completes the induction argument.

(3) Suppose that Assumption 1 (b) holds.

(3.a) We prove the claim by induction. Suppose that there is only one project left; that is, $s_t = 1$. If the project is assigned, then the surplus contributed by this project is $v - c_H$. Otherwise, since this is the last project, the Principal can get at most $\delta\big((v-c_L)(1-\alpha_H)+(v-c_H)\alpha_H\big)$. Because of the assumption, $q_t^*(c^{t-1}, c_H; s^t) = 1$.

Suppose that $q_{t'}^*(c^{t'}; s^{t'}) = 1$ for any $c^{t'} \neq c_H^{t'}$ such that $1 \leq s_{t'} \leq k-1$. Now we consider the case that there are $k \geq 1$ projects left ($s_t = k$). We assume that the Principal chooses some smallest integer $j \geq 1$ for $c^{t-1} \neq c_H^{t-1}$ such that $s_t = k$ and $q_{t+j-1}^*(c^{t-1}, c_H^j; s^{t+j-1}) = 1$. Let $W^j(c^{t-1}, c_H; s^t)$ be the future expected surplus in stage t for this particular choice of j. Then

$$W^j(c^{t-1}, c_H; s^t) - W^{j+1}(c^{t-1}, c_H; s^t) = \delta^{j-1}\alpha_H^{j-1}\big[W^1(c^{t-1}, c_H; s^t) - W^2(c^{t-1}, c_H; s^t)\big].$$

In particular,

$$W^1(c^{t-1}, c_H; s^t) = v - c_H + \delta \sum_{c_{t+1}} (v - c_{t+1}) P(c_{t+1}|c_H) + \cdots$$
$$+ \delta^{k-1} \sum_{c_{t+1},\ldots,c_{t+k-1}} (v - c_{t+k-1}) P(c_{t+1}|c_H) \prod_{2 \le i \le k-1} P(c_{t+i}|c_{t+i-1}),$$

and

$$W^2(c^{t-1}, c_H; s^t) = \delta \sum_{c_{t+1}} (v - c_{t+1}) P(c_{t+1}|c_H) + \cdots$$
$$+ \delta^{k} \sum_{c_{t+1},\ldots,c_{t+k}} (v - c_{t+k}) P(c_{t+1}|c_H) \prod_{2 \le i \le k} P(c_{t+i}|c_{t+i-1}).$$

In the calculation of $W^1(c^{t-1}, c_H; s^t)$ and $W^2(c^{t-1}, c_H; s^t)$, we have used the assumption that $q^*_{t'}(c^{t'}; s^{t'}) = 1$ for any $c^{t'} \neq c^{t'}_H$ such that $1 \le s_{t'} \le k-1$. To understand $W^1(c^{t-1}, c_H; s^t)$, the first term is the surplus in stage t since one project is assigned in this stage. Then there are $k-1$ projects left. Based on the induction hypothesis, the Principal shall continuously assign one project to the Agent in each subsequent stage. The term $W^2(c^{t-1}, c_H; s^t)$ can be understood similarly.

Then we have

$$W^1(c^{t-1}, c_H; s^t) - W^2(c^{t-1}, c_H; s^t)$$
$$= v - c_H - \delta^k \sum_{c_{t+1},\ldots,c_{t+k}} (v - c_{t+k}) P(c_{t+1}|c_H) \prod_{2 \le i \le k} P(c_{t+i}|c_{t+i-1}).$$

Using the inequality $v - c_H > \delta\big((v - c_L)(1 - \alpha_H) + (v - c_H)\alpha_H\big)$ iteratively, we have

$$v - c_H \ge \delta^k \sum_{c_{t+1},\ldots,c_{t+k}} (v - c_{t+k}) P(c_{t+1}|c_H) \prod_{2 \le i \le k} P(c_{t+i}|c_{t+i-1}).$$

which implies that $W^1(c^{t-1}, c_H; s^t) - W^2(c^{t-1}, c_H; s^t) > 0$. As a result, for any $j \ge 1$,

$$W^j(c^{t-1}, c_H; s^t) - W^{j+1}(c^{t-1}, c_H; s^t) > 0.$$

That is, $\{W^j(c^{t-1}, c_H; s^t)\}$ is a decreasing sequence in terms of j. If the Principal chooses $q^*_{t+j-1}(c^{t-1}, c^j_H; s^{t+j-1}) = 0$ for any $s_{t+j-1} = k$ and $j \ge 1$, then the future expected surplus is denoted by $W^\infty(c^{t-1}, c_H; s^t)$. It is obvious that $W^\infty(c^{t-1}, c_H; s^t)$ is the limit of the decreasing sequence $\{W^j(c^{t-1}, c_H; s^t)\}$, and hence is less than $W^1(c^{t-1}, c_H; s^t)$. As a result, the Principal should choose to assign the project immediately: $q^*_t(c^{t-1}, c_H; s^t) = 1$. The proof of this claim is thus completed.

(3.b) Again, we prove the claim by induction. In particular, when considering the Principal's payoff, we can ignore the surplus induced along the path with the initial cost being c_L.

We first consider the case that there is only one project last. Suppose that the first to the second-to-last units are assigned along the path $(c_H, c_H, \ldots)$ in stages $t_m < t_{m-1} < \ldots < t_2$, respectively. That is, if the cost is always high, then the Principal shall assign the first project in stage t_m, the second one in stage t_{m-1}, ..., and the second to last project in stage t_2. Note that we adopt the reverse order here to describe the assignment order.

If the Principal chooses $q_t^*(c_H^t; s^t) = 0$ for any $t > t_2$, then the surplus of E_P^m from this project is

$$\lambda_H \delta(v - c_L)(1 - \alpha_H) \sum_{k \geq t_2 - 1} \delta^k \alpha_H^k.$$

Suppose that the Principal chooses some $T > t_2$ and lets $q_t^*(c_H^t; s^t) = 0$ for $t_2 < t < T$ and $q_T^*(c_H^T; s^T) = 1$. The surplus of E_P^m from this project is

$$\begin{aligned} &\lambda_H \delta(v - c_L)(1 - \alpha_H) \sum_{k \geq t_2 - 1}^{T-2} \delta^k \alpha_H^k + \lambda_H \delta^{T-1} \alpha_H^{T-1}(v - c_H) \\ &- \lambda_L (c_H - c_L) \delta^{T-1} (\alpha_H - \alpha_L)^{T-1}. \end{aligned}$$

The latter is higher than the former if

$$v - c_H \geq \frac{\delta(v - c_L)(1 - \alpha_H)}{1 - \delta \alpha_H} + \frac{\lambda_L}{\lambda_H}(c_H - c_L) \left(\frac{\alpha_H - \alpha_L}{\alpha_H} \right)^{T-1},$$

which holds for sufficiently large T due to the assumption $v - c_H > \delta\big((v - c_L)(1 - \alpha_H) + (v - c_H)\alpha_H\big)$. Following the same argument as in the proof of Proposition 2 (3), the deadline T_1^{m*} of the last project is the same as that in the case $m = 1$.

In summary, if $T_1^{m*} > t_2$, then the Principal will assign the last project to the Agent before the deadline T_1^{m*} only when the report is c_L after a sequence of high costs, and assign the project regardless of the report in stage T_1^{m*}. If $T_1^{m*} \leq t_2$, the Principal shall assign the project to the Agent right after stage t_2.

Suppose that the claim is true for $r - 1$, and $T_1^{m*} \geq T_2^{m*} \geq \cdots \geq T_{r-1}^{m*}$, where T_j^{m*} is the deadline to assign the j-th to last project along the path $(c_H, c_H, \ldots)$ for $1 \leq j \leq r-1$. Now we consider the assignment rule of the r-th-to-last project. We take $t_m, \ldots, t_{r+1}$ as given.[11] The aim is to identify the deadline to assign the r-th-to-last project. We assume that $t_{r+1} < T_{r-1}^{m*} < \cdots < T_1^{m*}$ for simplicity.[12] There are three possibilities: the Principal assigns the project before the next deadline T_{r-1}^{m*}, or after that deadline, or never.

I. If the Principal decides to assign the r-th-to-last project in some stage i with $t_{r+1} < i < T_{r-1}^{m*}$ along the path $(c_H, c_H, \ldots)$, then the relevant surplus in E_P^m

[11] As in the previous case, if the cost is always high, then the Principal shall assign the first project in stage t_m, the second one in stage t_{m-1}, ..., and the $r+1$-th to last project in stage t_{r+1}.

[12] This is without loss of generality. Otherwise, the analysis below is the same by taking $T_{r-1}^{m*} = t_{r+1}+1$, and $T_j^{m*} = T_{j+1}^{m*} + 1$ for $1 \leq j \leq r-2$.

after stage t_{r+1} is

$$\begin{aligned}
&\lambda_H\delta(1-\alpha_H)\sum_{t_{r+1}-1\le k\le i-2}\delta^k\alpha_H^k\left[f_1(c_L)+\cdots+f_r(c_L)\right]\\
&+\lambda_H\delta^{i-1}\alpha_H^{i-1}(v-c_H)\\
&+\lambda_H\delta(1-\alpha_H)\sum_{i-1\le k\le T_{r-1}^{m*}-2}\delta^k\alpha_H^k\left[f_1(c_L)+\cdots+f_{r-1}(c_L)\right]\\
&+\lambda_H\delta^{T_{r-1}^{m*}-1}\alpha_H^{T_{r-1}^{m*}-1}(v-c_H)\\
&+\cdots\\
&+\lambda_H\delta(1-\alpha_H)\sum_{T_2^{m*}-1\le k\le T_1^{m*}-2}\delta^k\alpha_H^k f_1(c_L)\\
&+\lambda_H\delta^{T_1^{m*}-1}\alpha_H^{T_1^{m*}-1}(v-c_H)\\
&-\lambda_L(c_H-c_L)\left[\delta^{i-1}\left(\alpha_H-\alpha_L\right)^{i-1}+\sum_{1\le k\le r-1}\delta^{T_k^{m*}-1}\left(\alpha_H-\alpha_L\right)^{T_k^{m*}-1}\right].
\end{aligned}$$

The term $\lambda_H\delta(1-\alpha_H)\delta^k\alpha_H^k\left[f_1(c_L)+\cdots+f_r(c_L)\right]$ represents the expected surplus in the case that the r-th-to-last project is assigned upon a report c_L before stage i, and then the rest is assigned continuously in the subsequent stages. The term $\lambda_H\delta^{i-1}\alpha_H^{i-1}(v-c_H)$ is the expected payoff contributed by the r-th-to-last project in the case that the reports after stage t_{r+1} are always c_H until stage i. The term $\lambda_H\delta(1-\alpha_H)\delta^k\alpha_H^k\left[f_1(c_L)+\cdots+f_{r-1}(c_L)\right]$ describes the payoff in the case that the $r-1$-th-to-last project is assigned upon a report c_L after stage i, but before stage T_{r-1}^{m*}, and then the rest is assigned continuously in the subsequent stages. The term $\lambda_H\delta^{T_{r-1}^{m*}-1}\alpha_H^{T_{r-1}^{m*}-1}(v-c_H)$ is the expected payoff contributed by the $r-1$-th-to-last project in the case that the report is always c_H until the deadline T_{r-1}^{m*}. The other terms can be explained similarly. The last term is the rent left to the Agent.

II. Suppose that the Principal assigns the r-th-to-last project in stage $i\ge T_{r-1}^{m*}$. Let $T_0^{m*}=\infty$, and

$$I_j=\{i\colon i\ge T_{r-1}^{m*}, j \text{ is the largest } j' \text{ such that } j'\le r-2, T_{j'}^{m*}+j'>i+r-1\}$$

for $0\le j\le r-2$. If $i\in I_j$, then $i+1>T_{r-1}^{m*}$, and hence the $r-1$-th-to-last project will be assigned in stage $i+1$ immediately. The interpretation of the subscript j is that it is the first number that the j-th-to-last project could be assigned no later than its deadline. That is, $T_j^{m*}>i+(r-1-j)$, where i is the stage in which the r-th-to-last project is assigned, and $r-1-j$ is the total number of projects that are assigned after their deadlines because $i\ge T_{r-1}^{m*}$. Then $\{I_j\}_{0\le j\le r-2}$ is a partition of the sequence $\{T_{r-1}^{m*},T_{r-1}^{m*}+1,\ldots\}$ such that $i>i'$ for any $i\in I_j$ and $i'\in I_{j'}$ with $j<j'$.

If $i \in I_j$, then the payoff of the Principal after stage t_{r+1} is

$$\begin{aligned}
\Pi^r(j,i) = {} & \lambda_H \delta(1-\alpha_H) \sum_{t_{r+1}-1 \le k \le i-2} \delta^k \alpha_H^k \left[f_1(c_L) + \cdots + f_r(c_L)\right] \\
& + \lambda_H \delta^{i-1} \alpha_H^{i-1}(v - c_H) \\
& + \lambda_H \delta(1-\alpha_H) \delta^{i-1} \alpha_H^{i-1} \left[f_1(c_L) + \cdots + f_{r-1}(c_L)\right] \\
& + \lambda_H \delta^i \alpha_H^i (v - c_H) \\
& + \cdots \\
& + \lambda_H \delta(1-\alpha_H) \delta^{i+r-j-3} \alpha_H^{i+r-j-3} \left[f_1(c_L) + \cdots + f_{j+1}(c_L)\right] \\
& + \lambda_H \delta^{i+r-j-2} \alpha_H^{i+r-j-2} (v - c_H) \\
& + \lambda_H \delta(1-\alpha_H) \sum_{i+r-j-2 \le k \le T_j^{m*}-2} \delta^k \alpha_H^k \left[f_1(c_L) + \cdots + f_j(c_L)\right] \\
& + \lambda_H \delta^{T_j^{m*}-1} \alpha_H^{T_j^{m*}-1} (v - c_H) \\
& + \cdots \\
& + \lambda_H \delta(1-\alpha_H) \sum_{T_2^{m*}-1 \le k \le T_1^{m*}-2} \delta^k \alpha_H^k (v - c_H) \\
& + \lambda_H \delta^{T_1^{m*}-1} \alpha_H^{T_1^{m*}-1} (v - c_H) \\
& - \lambda_L (c_H - c_L) \Big[\sum_{0 \le k \le r-j-1} \delta^{i+k-1} \left(\alpha_H - \alpha_L\right)^{i+k-1} \\
& + \sum_{1 \le k \le j} \delta^{T_k^{m*}-1} \left(\alpha_H - \alpha_L\right)^{T_k^{m*}-1} \Big].
\end{aligned}$$

The term $\lambda_H \delta(1-\alpha_H)\delta^k \alpha_H^k \left[f_1(c_L) + \cdots + f_r(c_L)\right]$ represents the expected payoff in the case that the r-th-to-last project is assigned upon a report c_L before stage i, and then the rest is assigned continuously in the subsequent stages. The term $\lambda_H \delta^{i-1} \alpha_H^{i-1}(v - c_H)$ is the expected payoff contributed by the r-th-to-last project in the case that the report after stage t_{r+1} is always c_H until stage i. The term $\lambda_H \delta(1-\alpha_H)\delta^{i-1}\alpha_H^{i-1}\left[f_1(c_L) + \cdots + f_{r-1}(c_L)\right]$ describes the expected payoff in the case that the r-th-to-last project is assigned until stage i along the path $(c_H, c_H, \ldots)$ and then the report in the next stage is c_L. The other terms can be explained similarly. The last term is the rent left to the Agent.

III. If the Principal never assigns the r-th-to-last project along the path $(c_H, c_H, \ldots)$, then he gets the following expected payoff after stage t_{r+1},

$$\lambda_H \delta(1-\alpha_H) \sum_{k \ge t_{r+1}-1} \delta^k \alpha_H^k \left[f_1(c_L) + \cdots + f_r(c_L)\right].$$

The payoff in the case (III) is the limit of the payoff in the case (II) by taking i to the positive infinity.

We first consider case (I) above. To simplify the notation, we collect all the terms which do not depend on i in the payoff of case (I), and denote it by C_1.[13] The payoff in case (I) can be rewritten as

$$
\begin{aligned}
& C_1 - \lambda_H\delta(1-\alpha_H)f_r(c_L)\frac{\delta^{i-1}\alpha_H^{i-1}}{1-\delta\alpha_H} + \lambda_H\delta^{i-1}\alpha_H^{i-1}(v-c_H) \\
& - \lambda_L(c_H-c_L)\delta^{i-1}(\alpha_H-\alpha_L)^{i-1} \\
= & C_1 + \lambda_H\delta^{i-1}\alpha_H^{i-1}\big[(v-c_H) - \frac{\delta(1-\alpha_H)f_r(c_L)}{1-\delta\alpha_H} \\
& - \frac{\lambda_L}{\lambda_H}(c_H-c_L)\left(\frac{\alpha_H-\alpha_L}{\alpha_H}\right)^{i-1}\big].
\end{aligned}
$$

Following the same argument as in the proof of Proposition 2, the payoff in the above equation is increasing in i up to some point and then decreasing. In particular, the optimal stage $\tilde{t}_r$ to assign this project is the smallest t such that

$$
\begin{aligned}
(v-c_H) \geq & \ \delta\left((v-c_H)\alpha_H + f_r(c_L)(1-\alpha_H)\right) \\
& + \frac{\lambda_L}{\lambda_H}(c_H-c_L)\left[1-\delta(\alpha_H-\alpha_L)\right]\left[\frac{\alpha_H-\alpha_L}{\alpha_H}\right]^{t-1}.
\end{aligned}
$$

By Corollary 1, $f_r(c_L) \leq f_{r-1}(c_L)$, and hence $\tilde{t}_r \leq T^*_{r-1}$.

We then move to the case (II). Consider a hypothetical setting that there are only $r-1$ projects and the Principal assigns the $r-1$-th-to-last project in stage $i+1$. Let $\Pi^{r-1}(j, i+1)$ be the part of E_P^m which are contributed by these $r-1$ projects. Denote $C_2 = \lambda_H\delta(1-\alpha_H)f_r(c_L)\frac{\delta^{t_{r+1}-1}\alpha_H^{t_{r+1}-1}}{1-\delta\alpha_H}$.

Notice that

$$
\begin{aligned}
& \Pi^r(j,i) \\
= & \ \Pi^{r-1}(j, i+1) + \lambda_H\delta(1-\alpha_H)\sum_{t_{r+1}-1\leq k\leq i-2}\delta^k\alpha_H^k f_r(c_L)
\end{aligned}
$$

[13] In particular,

$$
\begin{aligned}
C_1 = & \ \lambda_H\delta(1-\alpha_H)f_r(c_L)\frac{\delta^{t_{r+1}-1}\alpha_H^{t_{r+1}-1}}{1-\delta\alpha_H} \\
& + \lambda_H\delta(1-\alpha_H)\sum_{t_{r+1}-1\leq k\leq T^*_{r-1}-2}\delta^k\alpha_H^k\left[f_1(c_L)+\cdots+f_{r-1}(c_L)\right] \\
& + \lambda_H\delta^{T^*_{r-1}-1}\alpha_H^{T^*_{r-1}-1}(v-c_H) \\
& + \cdots \\
& + \lambda_H\delta(1-\alpha_H)\sum_{T^*_2-1\leq k\leq T^*_1-2}\delta^k\alpha_H^k(v-c_H) \\
& + \lambda_H\delta^{T^*_1-1}\alpha_H^{T^*_1-1}(v-c_H) \\
& - \lambda_L(c_H-c_L)\sum_{1\leq k\leq r-1}\delta^{T^*_k-1}(\alpha_H-\alpha_L)^{T^*_k-1},
\end{aligned}
$$

which does not depend on i.

$$
\begin{aligned}
&+ \lambda_H \delta^{i-1}\alpha_H^{i-1}(v-c_H) - \lambda_L(c_H-c_L)\delta^{i-1}\left(\alpha_H-\alpha_L\right)^{i-1} \\
=\ &\Pi^{r-1}(j,i+1) \\
&+ \lambda_H\delta(1-\alpha_H)f_r(c_L)\frac{\delta^{t_{r+1}-1}\alpha_H^{t_{r+1}-1}}{1-\delta\alpha_H} - \lambda_H\delta(1-\alpha_H)f_r(c_L)\frac{\delta^{i-1}\alpha_H^{i-1}}{1-\delta\alpha_H} \\
&+ \lambda_H \delta^{i-1}\alpha_H^{i-1}(v-c_H) - \lambda_L(c_H-c_L)\delta^{i-1}\left(\alpha_H-\alpha_L\right)^{i-1} \\
=\ &\Pi^{r-1}(j,i+1) + C_2 \\
&+ \lambda_H\delta^{i-1}\alpha_H^{i-1}\left[(v-c_H) - \frac{\delta(1-\alpha_H)f_r(c_L)}{1-\delta\alpha_H} - \frac{\lambda_L}{\lambda_H}(c_H-c_L)\left(\frac{\alpha_H-\alpha_L}{\alpha_H}\right)^{i-1}\right].
\end{aligned}
$$

Due to our induction hypothesis and the fact that $i+1 > T^*_{r-1}$, $\Pi^{r-1}(j,i+1)$ must be less than the payoff contributed by the last $r-1$ projects via setting the deadline as the stage T^*_{r-1}. In addition, by the same argument in the case (I) above,

$$
\lambda_H\delta^{i-1}\alpha_H^{i-1}\big[v-c_H - \frac{\delta(1-\alpha_H)f_r(c_L)}{1-\delta\alpha_H} - \frac{\lambda_L}{\lambda_H}(c_H-c_L)\left(\frac{\alpha_H-\alpha_L}{\alpha_H}\right)^{i-1}\big]
$$

is increasing in terms of i up to a number (less than $\tilde{t}_r$, and hence also T^*_{r-1}), and then decreasing. As a result, the Principal's expected payoff in case (II) is less than that in case (I). The Principal should use the strategy in case (I), and hence set the deadline to assign the r-th-to-last project in stage $\tilde{t}_r$.[14] That is, $T^*_r = \tilde{t}_r$ is the smallest t such that

$$
\begin{aligned}
(v-c_H) \geq\ &\delta\left((v-c_H)\alpha_H + f_r(c_L)(1-\alpha_H)\right) \\
&+ \frac{\lambda_L}{\lambda_H}(c_H-c_L)\left[1-\delta(\alpha_H-\alpha_L)\right]\left[\frac{\alpha_H-\alpha_L}{\alpha_H}\right]^{t-1}.
\end{aligned}
$$

This completes the induction argument.

Step 2. Next, we construct the payment rule and check that the corresponding mechanism satisfies all the incentive constraints and participation constraints.

Given c^{t-1} and s^t with $s_t = 1$,

$$
U(c^{t-1}, c_L; s^t) = E^m_A(c^{t-1}; s^t) \text{ and } U(c^{t-1}, c_H; s^t) = 0.
$$

Let

$$
\begin{aligned}
p_t(c^{t-1}, c_t; s^t) = &\ U(c^{t-1}, c_t; s^t) + q_t(c^{t-1}, c_t; s^t)c_t \\
&- \delta q_t(c^{t-1}, c_t; s^t)\mathbb{E}(U(c^{t-1}, c_t, c_{t+1}; \tilde{s}^{t+1})|c_t) \\
&- \delta(1-q_t(c^{t-1}, c_t; s^t))\mathbb{E}(U(c^{t-1}, c_t, c_{t+1}; \hat{s}^{t+1})|c_t).
\end{aligned}
$$

Then all the participation constraints are satisfied and the Agent gets the payoff $U(c^{t-1}, c_t; s^t)$ at the cost c_t given c^{t-1} and s^t.

Following the argument in the proof of Lemma 3, the Agent with low cost would

[14] Since the payoff in case (III) is the limit of the payoff in case (II), the Principal will not choose the assignment rule in case (III).

not report high cost. We need to check that the Agent with high cost would not report low cost. Given c^{t-1} and s^t, if the Agent has cost c_H but reports c_L, then her payoff is

$$\begin{aligned}
&-q_t(c^{t-1}, c_L; s^t)c_H + p_t(c^{t-1}, c_L; s^t) \\
&+\delta q_t(c^{t-1}, c_L; s^t)\mathbb{E}\left[U(c^{t-1}, c_L, c_{t+1}; \tilde{s}^{t+1})|c_H\right] \\
&+\delta(1 - q_t(c^{t-1}, c_L; s^t))\mathbb{E}\left[U(c^{t-1}, c_L, c_{t+1}; \hat{s}^{t+1})|c_H\right] \\
=\,& U(c^{t-1}, c_L; s^t) - (c_H - c_L) \\
&-\delta\big[U(c^{t-1}, c_L, c_L; \tilde{s}^{t+1}) - U(c^{t-1}, c_L, c_H; \tilde{s}^{t+1})\big](\alpha_H - \alpha_L) \\
\leq\,& 0 \\
=\,& U(c^{t-1}, c_H; s^t).
\end{aligned}$$

The first equality holds since $q_t(c^{t-1}, c_L; s^t) = 1$. The inequality holds since

$$E_A^m(c^{t-1}; s^t) - \delta(\alpha_H - \alpha_L)E_A^m(c^{t-1}, c_L; \tilde{s}^{t+1}) \leq c_H - c_L.$$

This completes the proof. $\square$