Long Term Contracting under Limited Supply*

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Abstract

This paper studies the question how a revenue maximizing monopolist sells a finite copy of identical items to a buyer with changing values in a dynamic environment. The optimal selling mechanism is characterized via the allocation rules. It is shown that there is a deadline for each item such that the corresponding item cannot be allocated after its deadline. In particular, these deadlines do not depend on the initial stock, and the optimal mechanism in the setting under the capacity constraint may not converge to that in the setting without the capacity constraint. These features highlight the seller's inter-temporal trade-off on the opportunity cost.

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1 Introduction

Dynamic problems with limited supply are prevalent in many economic situations, and the capacity constraint could arise due to various reasons. A leading example is the sales of natural resources. A firm that is interested in maximizing its revenue by selling scarce resources sometimes has to face the constraint that the total supply is fixed, and known.[1] Since the market demand could fluctuate across time, it is typically not optimal for the firm to sell each unit of its output immediately after production/extraction. As a result, the firm needs to decide when and how to sell its product in a dynamic environment.

In this paper, we explore the question how a monopolist maximizes its revenue under the capacity constraint. In particular, we consider a dynamic setting in which a seller sells finite ($m \geq 1$) indivisible homogeneous items to a buyer in infinite horizon. In each stage, the buyer has unit demand and her private valuation is *high* or *low*, which evolves over time. The seller never learns the exact value, but he could require a report from the buyer. The buyer is strategic rather than truthful. We shall take a mechanism design approach; namely, the seller commits to a dynamic mechanism, which consists of a sequence of allocation rules and transfer rules that depend on the reporting history.

Since the seller has only finite items in stock, an immediate observation is that he faces a non-stationary environment. In particular, the number of items in stock could be different depending on whether the seller allocates one unit or not. This point is significantly different from that in the model without the capacity constraint, as the seller is facing the same stock at any stage in the latter environment. To be precise, in our setting, the seller needs to make an intertemporal trade-off on whether to allocate the current item based on a low value today, or to keep the item in order to wait for a possible high value in the future. This trade-off has two effects. First, postponing the allocation of the current item may delay the allocations of all the subsequent items. Second, if the seller allocates

[1]For instance, (1) an oil company that maximizes the revenue can sell oils up to a certain amount per day, while the total amount from an oil field is fixed; and (2) real estate developers or governments earn profits by selling the land use rights, but the total supply is usually limited (a government that sells the land use rights may have various concerns, and the revenue maximization is often one of the most important goals).

one item in the current stage, then he has to bear the opportunity cost that he is not able to allocate this item for a future high value. While the first feature also exists in the standard setting without the capacity constraint, the second feature is new in our setting, since the supply constraint makes the item *scarce*. As will be shown, this additional feature induces interesting properties on the allocation rules, and shapes the dynamics of the optimal mechanism accordingly.

The main result of this paper is to characterize the optimal selling mechanism via the allocation rules. In particular, we show that the seller shall fix a decreasing sequence of dates $T_m^* \leq T_{m-1}^* \leq \ldots \leq T_1^*$, where T_k^* is the allocation deadline of the k-th-to-last item. If the buyer has ever reported a high value, then all the items in stock will be allocated subsequently. However, if the buyer always reports the low value, then the seller will keep the k-th-to-last item until the deadline T_k^* is reached. The intuition behind these deadlines in the optimal mechanism is clear. On the one hand, the future payoff is discounted, and hence the seller will not wait forever. On the other hand, any delay not only postpones the allocation of the current item, but also may postpone the allocation of all the items in stock.

A surprising feature of this optimal mechanism is that the seller sets the deadlines in a memoryless way. That is, these deadlines do not depend on m, the initial number of items in stock. This result suggests when the seller considers a particular item, he cares about how many items need to be allocated in the *future*, rather than how many items have been allocated in the *past*.

We present two benchmark cases: one has no capacity constraint, and the other is a special case of our setting in which there is only one item in stock. In the former case, the seller sets a deadline T^* such that if the deadline has been reached, then he will allocate the items one by one in the subsequent stages regardless of the report.[2] When the capacity constraint is present, we show that T_∞^*, which is the limit of the decreasing sequence $\{T_m^*\}$, is (weakly) greater than T^*. Notably, this difference could be unbounded (depending on the parameters). As a result, the case with unlimited supply may not be a good approximation of the case that

[2]The case that the seller has unlimited supply is a variation of the model studied in Battaglini (2005). In the optimal mechanism, the seller will allocate one item to the buyer in every stage as long as the buyer has ever reported *high*. Along the lowest path with only report “low”, the seller sets a deadline T^* and allocates the item to the buyer in every stage regardless of the report once the deadline has been reached.

the seller has m items in stock even for sufficiently large m. The failure of this asymptotic property highlights the importance of the inter-temporal trade-off in our setting.

This paper contributes to the growing body of the literature on dynamic mechanism design.[3] For dynamic revenue-maximizing problems, the standard method is to work with the first-order approach; see Courty and Li (2000), Battaglini (2005), Board (2007), Esõ and Szentes (2007), and Kapička (2013) among others.[4] This approach, which generalizes the seminal work of Myerson (1981) to the dynamic setting, first identifies the necessary conditions for incentive compatibility so that one can work with a "relaxed problem" with the "local constraints", and then provides sufficient conditions to guarantee that the optimal allocation rule that solves this relaxed program also solves the original problem. A general treatment of this approach is provided in Pavan, Segal and Toikka (2014). The result in the current paper builds on the first-order approach. Our key difficulty is to identify the corresponding deadline for each item. In particular, we apply a backward induction argument to show that the deadlines are independent of the initial stock.

The paper also relates to an extensive literature that considers the dynamic environment in which the seller has a fixed number of goods and a deadline for selling the goods. The results in this literature usually study buyers who arrive over time and have private information for arrival/departure times, valuations or deadlines, but have no uncertainty towards their future valuations; see Pai and Vohra (2008), Gershkov and Moldovanu (2009, 2010), Said (2012), Board and Skrzypacz (2015), Hinnosaar (2015) and Mierendorff (2016) among others. An important exception is Ely, Garrett and Hinnosaar (2015). They study a two-stage dynamic mechanism design problem where the seller has limited capacity and buyers learn about their valuations over time, and show that overbooking

[3]For some earlier contributions, see, for example, Baron and Myerson (1982), Baron and Besanko (1984) and Besanko (1985). For further discussions, see the survey by Bergemann and Said (2011) and the recent symposium Bergemann and Pavan (2015).

[4]There are some exceptions. For example, Krähmer and Strausz (2015) study a sequential screening model under ex post participation constraint and do not adopt the standard first-order approach to sovle for the optimal mechanism. Recently, Garrett and Pavan (2015) and Battaglini and Lamba (2015) extend the analysis of the general model to settings where the first-order approach may not hold.

(selling more units than capacity) may be optimal. In contrast to all of these papers, in our environment, the capacity constraint is respected, the deadlines are endogenously generated in an infinite-horizon setting, and the buyer is uncertain about the future values.

The rest of the paper is organized as follows. We introduce the model in Section 2. Two benchmark cases are considered in Section 3. Section 4 presents the main result of the paper, and Section 5 concludes. The proof of the main result is collected in Section 6.

2 Model

We consider an infinitely repeated buyer-seller relationship in discrete time. The seller has $m \geq 1$ indivisible homogeneous items in stock and the buyer has unit demand in each stage. The items can be allocated in any period until they are sold out. Both buyer and seller are impatient and share a common discount factor $\delta \in (0, 1)$.

Value process. The buyer's value θ_1 at stage 1 is a random variable that takes either H or L ($0 < L < H$). The probability of drawing a high (resp. low) value is $\lambda(H)$ (resp. $\lambda(L) = 1 - \lambda(H)$). We assume that the distribution of the buyer's value tomorrow depends only on her value today, which implies that the stage-t value θ_t is a sufficient statistic for later values. In particular, we denote

$$P[\theta_{t+1} = H | \theta_t = H] = \alpha_H \quad \text{and} \quad P[\theta_{t+1} = H | \theta_t = L] = \alpha_L,$$

with $\alpha_H \geq \alpha_L$. That is, the process satisfies the first-order stochastic dominance assumption in the sense that the probability of *high* value tomorrow conditional on today's value being *high* first-order stochastically dominates the probability of *high* value tomorrow conditional on today's value being *low*. We adopt the following notation: a sequence of values from stage 1 to t is denoted by $\theta^t = (\theta_1, \ldots, \theta_t)$, and $\{\theta^0\} = \emptyset$.

Throughout this paper, we assume that $L > \delta\,(HP(H|L) + LP(L|L))$; that is, a low value in the current stage is more valuable than the discounted expected value in the next stage. This condition will be satisfied if the value "L" is sufficiently persistent or the discount factor is low.

Sales history. Let $s_1 = m$ indicate the fact that there are m items in stock at stage 1. If there are m' items ($m' \leq m$) in stock at stage t, then we denote $s_t = m'$. The sales history at stage t is characterized by the vector $s^t = (s_1, \ldots, s_t)$. In particular, if $s_t = m'$ and $s_{t+1} = m' - 1$, then it means that one item is allocated at stage t.

Mechansim. The seller can fully commit to a dynamic mechanism. By the revelation principle, we restrict our attention to incentive compatible direct mechanisms. A direct mechanism $\Gamma = (q, p)$ is a collection of allocation rules $q = \{q_t\}_{t\geq 1}$ and payments $p = \{p_t\}_{t\geq 1}$.

The timing is as follows:

1. At stage 1, the buyer randomly draws $\theta_1 \in \{H, L\}$ with probability $\lambda(\theta_1)$ and makes her report $\tilde{\theta}_1$. Based on the report, the seller allocates one unit with probability $q_1(\tilde{\theta}_1; s_1) \in [0, 1]$ and charges a transfer $p_1(\tilde{\theta}_1; s_1) \in \mathbb{R}$, where $s_1 = m$.

2. If one unit is allocated at stage t, then $s_{t+1} = s_t - 1$. Otherwise, $s_{t+1} = s_t$.

3. If $s_t \geq 1$ at stage $t > 1$, then the buyer draws a value θ_t following the law of motion $P[\theta_t|\theta_{t-1}]$ and makes her report $\tilde{\theta}_t$. Based on the reporting history $\tilde{\theta}^t = (\tilde{\theta}^{t-1}, \tilde{\theta}_t)$ and the sales history s^t, the allocation probability is $q_t(\tilde{\theta}^t; s^t) \in [0, 1]$ and a transfer $p_t(\tilde{\theta}^t; s^t) \in \mathbb{R}$ is charged.

4. If $s_t = 0$ for some t, then $q_t(\tilde{\theta}^t; s^t) \equiv 0$ and $p_t(\tilde{\theta}^t; s^t) \equiv 0$ for any $\tilde{\theta}^t$. This describes the situation that the seller has no item in stock and both parties leave the market.

The buyer's problem. Fix a mechanism $\Gamma = (q, p)$. Given the history θ^t up to stage $t \geq 0$, the sales history s^{t+1}, and the current value θ_{t+1} at stage $t + 1$, the buyer's expected payoff hereafter is

$$U(\theta^t, \theta_{t+1}; s^{t+1}) = \sum_{i \geq 1} \delta^{i-1} \mathbb{E}\big[\big(\theta_{t+i} q_{t+i}(\theta^{t+i}; s^{t+i}) - p_{t+i}(\theta^{t+i}; s^{t+i})\big) \,|\theta_{t+1}\big].$$

For any $t \geq 1$, the buyer is called *incentive compatible* at stage t (IC_t) if for any θ^{t-1}, θ_t, $\tilde{\theta}_t$, and s^t, we have

$$\begin{aligned}
&\theta_t q_t(\theta^t; s^t) - p_t(\theta^t; s^t) + \delta q_t(\theta^t; s^t) \mathbb{E}\left[U(\theta^t, \theta_{t+1}; \tilde{s}^{t+1}) | \theta_t\right] \\
&+ \delta(1 - q_t(\theta^t; s^t)) \mathbb{E}\left[U(\theta^t, \theta_{t+1}; \hat{s}^{t+1}) | \theta_t\right] \\
&\geq \theta_t q_t(\theta^{t-1}, \tilde{\theta}_t; s^t) - p_t(\theta^{t-1}, \tilde{\theta}_t; s^t) + \delta q_t(\theta^{t-1}, \tilde{\theta}_t; s^t) \mathbb{E}\left[U(\theta^{t-1}, \tilde{\theta}_t, \theta_{t+1}; \tilde{s}^{t+1}) | \theta_t\right] \\
&+ \delta(1 - q_t(\theta^{t-1}, \tilde{\theta}_t; s^t)) \mathbb{E}\left[U(\theta^{t-1}, \tilde{\theta}_t, \theta_{t+1}; \hat{s}^{t+1}) | \theta_t\right].
\end{aligned}$$

Notice that $\tilde{s}^{t+1} = (s^t, s_t - 1)$ describes the sales history that one item is allocated in the end of stage t and the stock at stage $t+1$ is $s_{t+1} = s_t - 1$, while $\hat{s}^{t+1} = (s^t, s_t)$ represents the sales history that no good is allocated at stage t and the stock at stage $t+1$ is still s_t. For $t \geq 1$, the buyer is said to be *individual rational* at stage t (IR_t) if $U(\theta^t; s^t) \geq 0$ for any θ^t and s^t. Notice that the IC_t and IR_t constraints will be automatically satisfied if $s_t = 0$.

The seller's problem. Given that the buyer truthfully reports all values, the expected revenue of the seller is

$$\sum_{t \geq 1} \delta^{t-1} \mathbb{E}\left[p_t(\theta^t; s^t)\right].$$

The seller's aim is to maximize the expected revenue subject to (IC_t) and (IR_t) for all $t \geq 1$.

3 Benchmark Cases

In this section, we shall discuss two benchmark cases. First, we consider the case that the seller has unlimited supply ($m = +\infty$). Second, we analyze the seller's problem for the case $m = 1$.

3.1 Unlimited supply

We shall study the seller's optimal allocation rule when $m = +\infty$.[5]

Let θ_L^t be the vector of values $(L, L, \ldots, L)$ with t elements; that is, θ_L^t is the lowest path with values being "L" from stage 1 to stage t. The following lemma shows that (1) the buyer's expected continuation payoff is zero when her value is low; and (2) the information rent for a high-value buyer depends on the allocation if her previous values remain low. This lemma is similar to that in Battaglini (2005), and its proof is omitted.

Lemma 1. *Suppose that q^* is the optimal allocation rule under all IC and IR constraints.*

1. *For any $t \geq 0$, θ^t and s^{t+1}, the expected continuation payoff of the buyer is $U(\theta^t, L; s^{t+1}) = 0$ when the current value is "L".*
2. *The expected payoff of the buyer with value "H" at stage 1 is*

$$U(H; s_1) = (H - L) \sum_{i=0}^{\infty} \delta^i \left(P(H|H) - P(H|L)\right)^i q_{i+1}^*(\theta_L^i, L; s^{i+1}).$$

3. *The expected revenue of the seller is:*

$$\sum_{\theta_1 \in \{H,L\}} \lambda(\theta_1) \left\{ q_1^*(\theta_1; s_1)\theta_1 + \delta \sum_{\theta_2 \in \{H,L\}} W(\theta_1, \theta_2; s^2) P(\theta_2|\theta_1) \right\}$$
$$- \lambda(H)(H - L) \sum_{i=0}^{\infty} \delta^i \left(P(H|H) - P(H|L)\right)^i q_{i+1}^*(\theta_L^{i+1}; s^{i+1}),$$

where W is the expected social welfare induced by the allocation rule q^*.

Notice that in Lemma 1 (3), the allocations in the last term of the seller's revenue only depend on the valuation history along the lowest path. As a result, the seller allocates one item as long as it is not on the lowest path. For the valuation history θ_L^{i+1}, the seller needs to compare $\lambda(L)L\delta^i P(L|L)^i$ and $\lambda(H)(H-L)\delta^i \left(P(H|H) - P(H|L)\right)^i$, which yields a deadline T^* as in the following proposition.

[5]As a discussed in the introduction, the result in this subsection is a variant of the result studied in Battaglini (2005). In Battaglini (2005), the seller's production cost is not zero and the buyer is not restricted to have the unit demand. However, all the arguments there still hold in the current setting.

Proposition 1. *The optimal mechanism is characterized by the following allocation rule.*

1. *For any θ^t and s^t, if $\theta^t \neq \theta^t_L$, then $q^*_t(\theta^t; s^t) = 1$.*
2. *Let T^* be the smallest t such that*

$$L \geq \frac{\lambda(H)}{\lambda(L)}(H-L)\left[\frac{P(H|H)-P(H|L)}{P(L|L)}\right]^{t-1}. \tag{1}$$

   Then

$$q^*_t(\theta^t_L; s^t) = \begin{cases} 0, & t < T^*; \\ 1, & t \geq T^*. \end{cases}$$

That is, (1) the seller allocates one unit as long as it is not on the lowest path, and (2) the seller will not allocate the item before the stage T^* along the lowest path, and will allocate one unit in each stage regardless of the history once the deadline T^* is reached.

3.2 One item: $m = 1$

In this subsection, we shall work with the simplest case of limited supply in the sense that $m = 1$. The main finding is that there is a deadline T^*_1 in the optimal mechanism. Within the deadline, the seller will allocate the item to the buyer as long as the buyer reports the high value, and keep the item if the reporting history is the lowest path. If the buyer has not reported "H" before the deadline, then the item will be allocated to the buyer at stage T^*_1 no matter what the buyer reports.

Lemma 2 below characterizes the expected payoffs of the buyer and the seller, which is an analogy of Lemma 1.

Lemma 2. *Suppose that q^* is the optimal allocation rule.*

1. *For any $t \geq 0$, θ^t and s^{t+1}, the expected payoff of the buyer is $U(\theta^t, L; s^{t+1}) = 0$ when the current value is "L".*
2. *The expected payoff of the buyer with value "H" at stage 1 is*

$$U(H; s_1) = (H-L)\sum_{i=0}^{\infty} \delta^i \left(P(H|H) - P(H|L)\right)^i q^*_{i+1}(\theta^{i+1}_L; s^{i+1}) \prod_{1\leq j\leq i} [1-q^*_j(\theta^j_L; s^j)].$$

3. *The expected revenue of the seller is:*

$$\sum_{\theta_1 \in \{H,L\}} \lambda(\theta_1)\big\{q_1^*(\theta_1;s_1)\theta_1 + \delta[1-q_1^*(\theta_1;s_1)]\big(W(\theta_1,H;s_1,s_1)P(H|\theta_1)$$
$$+ W(\theta_1,L;s_1,s_1)P(L|\theta_1)\big)\big\}$$
$$-\lambda(H)(H-L)\sum_{i=0}^{\infty}\delta^i \left(P(H|H)-P(H|L)\right)^i q_{i+1}^*(\theta_L^{i+1};s^{i+1})\prod_{1\le j\le i}[1-q_j^*(\theta_L^j;s^j)].$$

Remark 1. *Notice that the expected social welfare in Lemma 2 (3) above (the first summation) is argumented by the probability $[1-q_1^*(\theta_1;s_1)]$. That is, if $q_1^*(\theta_1;s_1)=1$, then the trade will end at this stage and the future values are irreverent (because there is only one item). If $q_1^*(\theta_1;s_1)<1$, then the trade in the future is relevant and the future expected social welfare needs to be adjusted accordingly.*

The following proposition characterizes the optimal allocation rule for the one-item case.

Proposition 2. *The optimal mechanism is characterized by the following allocation rule.*

1. *For any θ^t and s^{t+1} such that the item is still in stock $(s_{t+1}=1)$, if $\theta_{t+1}=H$, then $q_{t+1}^*(\theta^t,H;s^{t+1})=1$.*
2. *Let T_1^* be the smallest t such that*

$$\begin{aligned} L \quad & \ge \delta\left(HP(H|L)+LP(L|L)\right) \\ & + \frac{\lambda(H)}{\lambda(L)}(H-L)\left[1-\delta(P(H|H)-P(H|L))\right]\left[\frac{P(H|H)-P(H|L)}{P(L|L)}\right]^{t-1} \end{aligned} \tag{2}$$

Then

$$q_t^*(\theta_L^t;s^t)=\begin{cases}0, & t<T_1^*;\\ 1, & t=T_1^*.\end{cases}$$

That is, the seller will not allocate the item before the deadline T_1^* along the lowest path, and will allocate the item at stage T_1^* regardless of the report.

If the value "H" is not persistent, then the optimal deadline T_1^* must be finite.

In particular, as $t \to \infty$, the right-hand side of inequality (2) approaches

$$\delta \left(HP(H|L) + LP(L|L) \right),$$

which implies that inequality (2) is satisfied for sufficiently large t.

Property (1) in Proposition 2 is obvious, the main task is to prove property (2). That is, we need to find an optimal stopping (finite or infinite) stage for the seller to allocate the item to the buyer who reports the low value all the time. By our assumption $L > \delta \left(HP(H|L) + LP(L|L) \right)$, a low value in the current stage is more valuable than the discounted expected value in the next stage. As a result, the optimal stopping time is always finite. We then show that the seller's revenue generated by a particular choice of the stopping time t' is first increasing, and then decrease in terms of t'. Thus, an optimal stopping stage exists and is finite. The details of the proof is left in the end of this subsection.

In the following corollary, we summarize two interesting special cases. In the first case, there is no serial correlation. As a result, the value today cannot convey any information for the value tomorrow. Since low value today is more valuable than the discounted expected value tomorrow, the seller will allocate the item immediately. In the second case, we assume the high value to be perfectly persistent, and it turns out that the seller will allocate the item along the lowest path either immediately or never (depending on the parameters).

Corollary 1. *1. No serial correlation:* $P(H|H) = P(H|L)$. *The seller allocates the item immediately at stage 1:* $q_1^*(\theta_1; s_1) = 1$ *for any* θ_1.

2. Persistent high value: $P(H|H) = 1$. *The optimal allocation rule is as follows.*

(a) If

$$L < \delta \left(HP(H|L) + LP(L|L) \right) + \frac{\lambda(H)}{\lambda(L)} (H - L)(1 - \delta P(L|L)),$$

then the seller will allocate the item only when receiving the report "H".

(b) If

$$L \geq \delta \left(HP(H|L) + LP(L|L) \right) + \frac{\lambda(H)}{\lambda(L)} (H - L)(1 - \delta P(L|L)),$$

then the seller will allocate the item at stage 1 *regardless of the report.*

Below, we provide the proof of Proposition 2.

Proof of Proposition 2. By Lemma 2, the expected revenue of the seller is:

$$
\begin{aligned}
&E_R \\
&= \sum_{\theta \in \{H,L\}} \lambda(\theta)\big\{q_1^*(\theta; s_1)\theta \\
&+ \delta[1 - q_1^*(\theta; s_1)](W(\theta, H; s_1, s_1)P(H|\theta) + W(\theta, L; s_1, s_1)P(L|\theta))\big\} \\
&- \lambda(H)(H-L)\sum_{i=0}^{\infty} \delta^i \left(P(H|H) - P(H|L)\right)^i q_{i+1}^*(\theta_L^{i+1}; s^{i+1}) \prod_{1\le j\le i} [1 - q_j^*(\theta_L^j; s^j)].
\end{aligned}
\tag{3}
$$

Notice that the seller's revenue E_R is linear in q_t^* for any t. Thus, q_t^* can be either 1 or 0 for each t, which implies that there is a deterministic optimal allocation rule. An immediate consequence of this observation is that only one term in the summation

$$
\lambda(H)(H-L)\sum_{i=0}^{\infty} \delta^i \left(P(H|H) - P(H|L)\right)^i q_{i+1}^*(\theta_L^i, L; s^{i+1}) \prod_{1\le j\le i} [1 - q_j^*(\theta_L^j; s^j)]
$$

is relevant for the seller's revenue.

(1) This case is obvious. For any θ^t and s^{t+2} such that $s_{t+1} = 1$, since

$$
H > \delta(W(\theta^t, H, H; s^{t+2})P(H|H) + W(\theta^t, H, L; s^{t+2})P(L|H)),
$$

we have that $q_{t+1}^*(\theta^t, H; s^{t+1}) = 1$, which implies that the trade will end as long as the report is "H".

(2) Suppose that the seller chooses some $t' \ge 1$, and lets $q_t^*(\theta_L^t; s^t) = 0$ for any $t < t'$ and $q_{t'}^*(\theta_L^{t'}; s^{t'}) = 1$. Notice that no trade will occur after stage t' since the seller will allocate the good at stage t' regardless of the report. For simplicity, we can ignore the social welfare in Equation (3) which is from the path that the initial

value is “H”,[6] and focus on the following part:

$$\begin{aligned}
&\lambda(L)\delta \sum_{k\geq 0}^{t'-2}[\delta^k P(L|L)^k]P(H|L)H + \lambda(L)\delta^{t'-1}P(L|L)^{t'-1}L \\
&- \lambda(H)(H-L)\delta^{t'-1}\left(P(H|H)-P(H|L)\right)^{t'-1} \\
&= \lambda(L)[1-\delta^{t'-1}P(L|L)^{t'-1}]\frac{\delta P(H|L)H}{1-\delta P(L|L)} + \lambda(L)\delta^{t'-1}P(L|L)^{t'-1}L \\
&- \lambda(H)(H-L)\delta^{t'-1}\left(P(H|H)-P(H|L)\right)^{t'-1}. \qquad (4)
\end{aligned}$$

The first term describes the expected social welfare in the case that the initial value is “L” and it changes to “H” before stage $t'+1$. The second term is the part of the expected social welfare for the case that the values in the previous t' stages are “L”. The last term is the rent left to the buyer.

If the seller chooses never to allocate the good along the lowest path (i.e., $q_t^*(\theta_L^t; s^t) = 0$ for any t), then the relevant part in the seller's revenue is

$$\lambda(L)\delta \sum_{k\geq 0}[\delta^k P(L|L)^k]P(H|L)H = \lambda(L)\frac{\delta P(H|L)H}{1-\delta P(L|L)}, \qquad (5)$$

where the term $\lambda(L)\delta^{k+1}P(L|L)^k P(H|L)H$ is the expected social welfare that the value is L until stage $k+1$ and the value is H at stage $k+2$.

The difference of the revenues for these two allocation rules is

$$\begin{aligned}
(5)-(6) &= -\lambda(L)\delta^{t'-1}P(L|L)^{t'-1}\frac{\delta P(H|L)H}{1-\delta P(L|L)} + \lambda(L)\delta^{t'-1}P(L|L)^{t'-1}L \\
&- \lambda(H)(H-L)\delta^{t'-1}\left(P(H|H)-P(H|L)\right)^{t'-1} \\
&= \lambda(L)\delta^{t'-1}P(L|L)^{t'-1}\big[L - \frac{\delta P(H|L)H}{1-\delta P(L|L)} \\
&- \frac{\lambda(H)}{\lambda(L)}(H-L)\left(\frac{P(H|H)-P(H|L)}{P(L|L)}\right)^{t'-1}\big].
\end{aligned}$$

[6] Since the social welfare from the path that the initial value is “H” in Equation (3) is not influenced by the allocation rule along the lowest path, we can ignore it when we compare the revenues of the seller from different allocation rules along the lowest path.

This difference is positive if

$$L - \frac{\delta P(H|L)H}{1-\delta P(L|L)} \geq \frac{\lambda(H)}{\lambda(L)}(H-L)\left(\frac{P(H|H)-P(H|L)}{P(L|L)}\right)^{t'-1},$$

which holds for sufficiently large t'. As a result, the seller will choose to allocate the good at some (finite) date.

Denote $a = \frac{\lambda(L)}{\delta P(L|L)}\left(L - \frac{\delta P(H|L)H}{1-\delta P(L|L)}\right)$, $b = \delta P(L|L)$, $c = \frac{\lambda(H)}{\delta(P(H|H)-P(H|L))}(H-L)$, $d = \delta(P(H|H)-P(H|L))$. Define a function as $f(t) = ab^t - cd^t$. Taking the first order derivative of f, we get $f'(t) = (a\ln b)b^t - (c\ln d)d^t$. Then $f'(t) \geq 0$ if and only if $t \leq \ln(\frac{a\ln b}{c\ln d})/\ln\frac{d}{b}$. As a result, f is increasing until $\ln(\frac{a\ln b}{c\ln d})/\ln\frac{d}{b}$ and then decreasing. Notice that (5) $-$ (6) $= ab^{t'} - cd^{t'}$, and the seller's revenue in (4) is the summation of $ab^{t'} - cd^{t'}$ and the constant $\lambda(L)\frac{\delta P(H|L)H}{1-\delta P(L|L)}$. Thus, the seller's revenue is also increasing until $\ln(\frac{a\ln b}{c\ln d})/\ln\frac{d}{b}$ and then decreasing.

To identify the optimal date to allocate the item along the lowest path, we need to find the smallest t' such that

$$\begin{aligned}&\lambda(L)\delta^{t'-1}P(L|L)^{t'-1}\big[L - \frac{\delta P(H|L)H}{1-\delta P(L|L)} - \frac{\lambda(H)}{\lambda(L)}(H-L)\left(\frac{P(H|H)-P(H|L)}{P(L|L)}\right)^{t'-1}\big]\\ &\geq \lambda(L)\delta^{t'}P(L|L)^{t'}\big[L - \frac{\delta P(H|L)H}{1-\delta P(L|L)} - \frac{\lambda(H)}{\lambda(L)}(H-L)\left(\frac{P(H|H)-P(H|L)}{P(L|L)}\right)^{t'}\big].\end{aligned}$$

That is,

$$\begin{aligned}&L - \frac{\delta P(H|L)H}{1-\delta P(L|L)} - \frac{\lambda(H)}{\lambda(L)}(H-L)\left(\frac{P(H|H)-P(H|L)}{P(L|L)}\right)^{t'-1}\\ &\geq \delta P(L|L)\big[L - \frac{\delta P(H|L)H}{1-\delta P(L|L)} - \frac{\lambda(H)}{\lambda(L)}(H-L)\left(\frac{P(H|H)-P(H|L)}{P(L|L)}\right)^{t'}\big],\end{aligned}$$

which implies that

$$\begin{aligned}L &\geq \delta\left(HP(H|L) + LP(L|L)\right)\\ &+ \frac{\lambda(H)}{\lambda(L)}(H-L)\left[1-\delta(P(H|H)-P(H|L))\right]\left[\frac{P(H|H)-P(H|L)}{P(L|L)}\right]^{t'-1}.\end{aligned}$$

The proof is complete. $\square$

4 Main Results

In this section, we shall analyze the seller's problem subject to a general capacity constraint ($m \geq 1$). We show that the seller will set a sequence of dates $T_m^* \leq T_{m-1}^* \leq \ldots \leq T_1^*$ such that T_k^* is the deadline for the current item when there are k items in stock. It is proved that (1) each deadline T_k^* is independent of the initial stock (i.e., T_k^* is the same for any m); and (2) the sequence $\{T_k^*\}$ is decreasing.

We compare our results with the benchmark case of unlimited supply as considered in Section 3. It is shown that there is some *dissonance* between the case of unlimited supply and the limit of the result in this section by taking m to the positive infinity. In particular, we show that the inter-temporal trade-off plays an important role due to the presence of the supply constraint, in the sense that the seller is more willing to wait for a possible future high value "H" rather than selling the item based on the current report "L" in the setting here.

Let

$$f_k(H) = \delta^{k-1} \sum_{\theta_2 \in \{H,L\},\ldots,\theta_k \in \{H,L\}} \left(P(\theta_2|H)\theta_k \cdot \prod_{2 \leq j \leq k-1} P(\theta_{j+1}|\theta_j) \right)$$

and

$$f_k(L) = \delta^{k-1} \sum_{\theta_2 \in \{H,L\},\ldots,\theta_k \in \{H,L\}} \left(P(\theta_2|L)\theta_k \cdot \prod_{2 \leq j \leq k-1} P(\theta_{j+1}|\theta_j) \right).$$

Note that $f_1(H) = H$ and $f_1(L) = L$. The function $f_k(\theta)$ is the discounted expected value at stage k when the initial value is θ for $k \geq 1$ and $\theta \in \{H, L\}$. The following lemma is obvious.

Lemma 3. *For any $\theta \in \{H, L\}$, $\{f_k(\theta)\}$ is a decreasing sequence and converges to 0 as k increases.*

Proof. Recall that

$$L > \delta \left(HP(H|L) + LP(L|L) \right).$$

It is also obvious that

$$H > \delta \left(HP(H|H) + LP(L|H) \right).$$

By iterating these two inequalities, it can be easily checked that $\{f_k(\theta)\}$ is a decreasing sequence. As $k \to \infty$, δ^k converges to 0, which implies that $f_k(\theta)$ converges to 0 for any $\theta \in \{H, L\}$. $\square$

The following theorem characterizes the optimal mechanism via the allocation rule.

Theorem 1. *The optimal mechanism is characterized by the following allocation rule.*

1. *For any θ^t and s^{t+1} such that $s_{t+1} \geq 1$, $q^*_{t+1}(\theta^t, H; s^{t+1}) = 1$.*
2. *For any $\theta^t \neq \theta^t_L$ and s^{t+1} such that $s_{t+1} \geq 1$, $q^*_{t+1}(\theta^t, L; s^{t+1}) = 1$.*
3. *For $1 \leq k \leq m$, let T^*_k be the smallest t such that*

$$\begin{aligned} L \quad & \geq \delta\left(LP(L|L) + f_k(H)P(H|L)\right) \\ & + \frac{\lambda(H)}{\lambda(L)}(H - L)\left[1 - \delta(P(H|H) - P(H|L))\right]\left[\frac{P(H|H) - P(H|L)}{P(L|L)}\right]^{t-1} \end{aligned} \tag{6}$$

Then for $\theta^t = \theta^t_L$ and s^t such that $s_t = k \geq 1$,

$$q^*_t(\theta^t_L; s^t) = \begin{cases} 0, & t < T^*_k; \\ 1, & t \geq T^*_k. \end{cases}$$

That is, the seller with k items in stock will not allocate the current item along the lowest path before the deadline T^*_k, but will allocate this unit immediately as long as the stage T^*_k is reached.

As in Lemmas 1 and 2, the seller's revenue can be pinned down as a function which depends on the difference of the social welfare and the rent left to the buyer. The main difficulty is to identify the corresponding deadline for each item. The analysis of the last item is similar as that in the one-item benchmark case considered in Subsection 3.2. The argument is more involved for the k-th-to last item ($k > 1$). We adopt a backward induction argument to prove inequality (6). The proof is left in Section 6.

The following corollary summarizes two important features of the deadlines.

Corollary 2. 1. *For each k, T_k^* does not depend on the number of the original stock m, and $T_1^* \geq T_2^* \geq \cdots$.*

2. *Let T_∞^* be the smallest t such that*

$$L \geq \frac{\lambda(H)}{\lambda(L)}(H-L)\left[\frac{1-\delta(P(H|H)-P(H|L))}{1-\delta P(L|L)}\right]\left[\frac{P(H|H)-P(H|L)}{P(L|L)}\right]^{t-1}.$$

Then T_∞^* is the limit of the decreasing sequence $\{T_k^*\}$. We have that $T_\infty^* \geq T^*$, and the difference could be unbounded.

Proof. (1) By inequality (6), it is obvious that T_k^* does not depend on m for each k. The sequence is decreasing since $\{f_k(H)\}$ is decreasing in k (see Lemma 3).

(2) Recall that T^* is the smallest t such that

$$L \geq \frac{\lambda(H)}{\lambda(L)}(H-L)\left[\frac{P(H|H)-P(H|L)}{P(L|L)}\right]^{t-1}.$$

Then $T_\infty^* \geq T^*$ since

$$\frac{P(H|H)-P(H|L)}{P(L|L)} \leq 1 \text{ and } \frac{1-\delta(P(H|H)-P(H|L))}{1-\delta P(L|L)} \geq 1,$$

Notice that the definition of T^* does not depend on the discount factor δ, but T_∞^* does depend on δ. In the extreme case that $P(L|L) = 1$ but $P(H|H) < 1$, taking $\delta \to 1$ will not change T^*, but will force T_∞^* to go to the positive infinity. As a result, in the case that is sufficiently close to this extreme case, the gap between T^* and T_∞^* could be very large. □

Remark 2. *When the seller has only limited supply, he has to deal with the intertemporal trade-off on the opportunity cost that allocating one item today with low value may decrease the revenue due to the possible high value tomorrow, while waiting for the high value tomorrow could delay the allocations of all the items in stock. When the discount factor is high, the seller is willing to delay the allocations of the items since the cost to do so is low. Importantly, even though the initial stock m could be very large, the supply constraint always makes the item scarce. As a result, the seller could be willing to wait for a very long time to allocate the first item. Such kind of scarcity does not exist when the seller has infinite supply.*

5 Conclusion

In this paper, we address the question how a revenue maximizing monopolist sells a finite copy of identical items to a buyer with changing values in a dynamic environment. We provide a characterization for the optimal selling mechanism via the allocation rules. It is shown that there is a deadline for each item such that the corresponding item cannot be allocated after its deadline. Two important features of the optimal mechanism deserve to be emphasized. First, these deadlines do not depend on the initial stock. Second, the optimal mechanism in the setting under the capacity constraint may not converge to that in the setting without the capacity constraint even if we increase the number of the initial stock. These features highlight the seller's trade-off on the opportunity cost. In reality, the seller often cannot produce unlimited product due to various reasons, such as the limit on the input of capital/labor/land. In this sense, this paper takes one step in understanding such situation.

6 Proof of Theorem 1

The following lemma is an analogy of Lemma 2, which is more complicated. The proof is similar to that in Battaglini (2005), and hence is omitted.

Lemma 4. *Suppose that q^* is the optimal allocation rule.*

1. *For any $t \geq 0$, θ^t and s^{t+1}, the expected payoff of the buyer is $U(\theta^t, L; s^{t+1}) = 0$ when the current value is "L".*
2. *The expected payoff of the buyer with value "H" at stage 1 is*

$$
\begin{aligned}
& U(H; s_1) \\
&= (H-L)\sum_{i_1=0}^{\infty} \delta^{i_1} \left(P(H|H) - P(H|L)\right)^{i_1} q^*_{i_1+1}(\theta_L^{i_1+1}; s^{i_1+1}) \prod_{1\leq j\leq i_1} [1 - q^*_j(\theta_L^j; s^j)] \\
&+ (H-L)\sum_{\substack{i_1\geq 0\\ i_2\geq 1}} \{\delta^{i_1+i_2} \left(P(H|H) - P(H|L)\right)^{i_1+i_2} q^*_{i_1+1}(\theta_L^{i_1+1}; s^{i_1+1}) \\
&\cdot q^*_{i_1+i_2+1}(\theta_L^{i_1+i_2+1}; s^{i_1+i_2+1}) \prod_{\substack{1\leq j\leq i_1+i_2\\ j\neq i_1+1}} [1 - q^*_j(\theta_L^j; s^j)]\}
\end{aligned}
$$

$$
\begin{aligned}
&+\cdots \\
&+(H-L)\sum_{\substack{i_1\geq 0\\ i_2\geq 1\\ \cdots\\ i_m\geq 1}}\big\{\delta^{\sum_{k=1}^m i_k}\left(P(H|H)-P(H|L)\right)^{\sum_{k=1}^m i_k} \\
&\cdot\prod_{k=1}^m q^*_{\sum_{l=1}^k i_l+1}(\theta_L^{\sum_{l=1}^k i_l+1};s^{\sum_{l=1}^k i_l+1})\prod_{\substack{1\leq j\leq\sum_{k=1}^m i_k\\ j\neq i_1+1\\ j\neq i_1+i_2+1\\ \cdots\\ j\neq\sum_{k=1}^{m-1}i_k+1}}[1-q^*_j(\theta_L^j;s^j)]\big\}.
\end{aligned}
$$

In the above equation, s^{i_1+1} is the constant vector with i_1+1 components m, indicating that the first item has not been allocated until stage i_1+1. Similarly, $s^{i_1+i_2+1}$ is the vector with i_1+1 components m and i_2 components $m-1$, implying that the first item is allocated at stage i_1+1 and the second item has not been allocated until stage i_1+i_2+1. The other terms can be explained similarly.

3. *The expected revenue of the seller is:*

$$
\begin{aligned}
&\sum_{\theta_1\in\{H,L\}}\lambda(\theta_1)\big\{q_1^*(\theta_1;s_1)\theta_1 \\
&+\delta q_1^*(\theta_1;s_1)\left(W(\theta_1,H;s_1,s_1-1)P(H|\theta_1)+W(\theta_1,L;s_1,s_1-1)P(L|\theta_1)\right) \\
&+\delta[1-q_1^*(\theta_1;s_1)]\left(W(\theta_1,H;s_1,s_1)P(H|\theta_1)+W(\theta_1,L;s_1,s_1)P(L|\theta_1)\right)\big\} \\
&-\lambda(H)U(H;s_1).
\end{aligned}
$$

Remark 3. *Though the buyer's expected payoff in Lemma 4 (2) is complicated, its intuition is clear. The first term is the expected rent pinned down to the lowest path until the stage at which the first item is allocated. The k-th term corresponds to the expected rent pinned down to the lowest path between the stage at which the k-th item is allocated and the stage at which the $k+1$-th item is allocated.*

In Lemma 4 (3), we need to distinguish the case that one unit is allocated at this stage and the case that the seller does not allocate this item. In the first case, the future expected social welfare is represented by $W(\theta_1,\theta_2;s_1,s_1-1)$, which means that one good is allocated at stage 1 and hence the total number of stock is decreased by one unit. In the second case, no good has been allocated at stage 1 and the total *number of stock keeps unchanged.*

Proof of Theorem 1. The following intuition from the proof of Proposition 2 still holds: the seller's revenue E_R is linear in q_t^* for any t. Thus, the optimal allocation rule can be deterministic.

(1) This case is obvious. Suppose that at stage t there are k goods left, and $\theta_t = H$. No matter whether one good is allocated or not at this stage, the seller can always adopt the same selling strategy for the next $k-1$ goods.

1. If one good is allocated at stage t, then the $k-1$ goods are all the goods left.
2. If no good is allocated at this stage, then the $k-1$ goods are the k-th-to-last item to the second-to-last item.

The welfare contributed by these $k-1$ goods are the same in these two situations. In addition, the welfare contributed by the good allocated at stage t in case (1) is greater than the welfare contributed by the last good in case (2). As a result, the seller will allocate one unit as long as the report is "H".

(2) We prove this claim by induction.

Suppose that there is only one item left; that is, $s_{t+1} = 1$. If the good is allocated, then the seller gets "L". Otherwise, since this is the last good, the seller can get at most $\delta(HP(H|L) + LP(L|L))$. Because $L > \delta\left(HP(H|L) + LP(L|L)\right)$, $q^*_{t+1}(\theta^t, L; s^{t+1}) = 1$.

Suppose that there are $k \geq 1$ goods left ($s_{t+1} = k$), and $q^*_{t'}(\theta^{t'}; s^{t'}) = 1$ for any $\theta^{t'} \neq \theta_L^{t'}$ such that $s_{t'} \leq k-1$. We assume that the seller chooses some smallest integer $j \geq 1$ for $\theta^t \neq \theta_L^t$ such that $s_{t+j} = k$ and $q^*_{t+j}(\theta^{t+j-1}, L; s^{t+j}) = 1$. Let $W^j(\theta^t, L; s^{t+1})$ be the future expected social welfare at date $t+1$ for this allocation rule. It is easy to see that

$$W^j(\theta^t, L; s^{t+1}) - W^{j+1}(\theta^t, L; s^{t+1}) = \delta^{j-1} P(L|L)^{j-1}\left[W^1(\theta^t, L; s^{t+1}) - W^2(\theta^t, L; s^{t+1})\right].$$

In particular,

$$W^1(\theta^t, L; s^{t+1}) = L + \delta \sum_{\theta_{t+2}} \theta_{t+2} P(\theta_{t+2}|L) + \cdots$$

$$+\,\delta^{k-1}\sum_{\theta_{t+2},\ldots,\theta_{t+k}}\theta_{t+k}P(\theta_{t+2}|L)\prod_{2\le i\le k-1}P(\theta_{t+i+1}|\theta_{t+i}),$$

and

$$\begin{aligned}W^2(\theta^t,L;s^{t+1}) &= \delta\sum_{\theta_{t+2}}\theta_{t+2}P(\theta_{t+2}|L)+\cdots\\ &\quad+\delta^k\sum_{\theta_{t+2},\ldots,\theta_{t+k+1}}\theta_{t+k+1}P(\theta_{t+2}|L)\prod_{2\le i\le k}P(\theta_{t+i+1}|\theta_{t+i}).\end{aligned}$$

As a result,

$$W^1(\theta^t,L;s^{t+1})-W^2(\theta^t,L;s^{t+1}) = L-\delta^k\sum_{\theta_{t+2},\ldots,\theta_{t+k+1}}\theta_{t+k+1}P(\theta_{t+2}|L)\prod_{2\le i\le k}P(\theta_{t+i+1}|\theta_{t+i}).$$

Using the inequality $L>\delta\left(HP(H|L)+LP(L|L)\right)$ iteratively, we have

$$L>\delta^k\sum_{\theta_{t+2},\ldots,\theta_{t+k+1}}\theta_{t+k+1}P(\theta_{t+2}|L)\prod_{2\le i\le k}P(\theta_{t+i+1}|\theta_{t+i}).$$

As a result, $W^1(\theta^t,L;s^{t+1})-W^2(\theta^t,L;s^{t+1})>0$, which implies that for any $j\ge 1$,

$$W^j(\theta^t,L;s^{t+1})-W^{j+1}(\theta^t,L;s^{t+1})>0.$$

That is, $\{W^j(\theta^t,L;s^{t+1})\}$ is a decreasing sequence.

In addition, if the seller chooses $q^*_{t+j}(\theta^{t+j-1},L;s^{t+j})=0$ for any $s_{t+j}=k$ and $j\ge 1$, then the future expected social welfare is denoted by $W^\infty(\theta^t,L;s^{t+1})$. It is obvious that $W^\infty(\theta^t,L;s^{t+1})$ is the limit of the decreasing sequence $\{W^j(\theta^t,L;s^{t+1})\}$, and hence is less than $W^1(\theta^t,L;s^{t+1})$. As a result, $q^*_{t+1}(\theta^t,L;s^{t+1})=1$. The proof of this claim is thus completed.

(3) We prove the claim by induction. In particular, when considering the seller's revenue, we shall ignore the social welfare induced along the path that the initial state is "H" as it does not affect our analysis.

We first consider the allocation rule for the last good. Suppose that the first to the second-to-last units are allocated along the lowest path at stages $t_m<t_{m-1}<\ldots<t_2$, respectively. That is, $q^*_j(\theta^j_L;s^j)=0$ for $1\le j\le t_2$ and $j\neq$

$t_m, t_{m-1}, \ldots, t_2$, and $q_j^*(\theta_L^j; s^j) = 1$ for $j = t_m, t_{m-1}, \ldots, t_2$. If the seller chooses $q_t^*(\theta_L^t; s^t) = 0$ for any $t > t_2$, then the relevant part of the seller's revenue, which depends on this allocation rule, is

$$\lambda(L)\delta H P(H|L) \sum_{k \geq t_2 - 1} \delta^k P(L|L)^k.$$

Suppose that the seller chooses some $T > t_2$ and lets $q_t^*(\theta_L^t; s^t) = 0$ for $t_2 < t < T$ and $q_T^*(\theta_L^T; s^T) = 1$. The relevant part of the seller's revenue is

$$\lambda(L)\delta H P(H|L) \sum_{k \geq t_2 - 1}^{T-2} \delta^k P(L|L)^k + \lambda(L)\delta^{T-1} P(L|L)^{T-1} L$$
$$- \lambda(H)(H - L)\delta^{T-1} \left(P(H|H) - P(H|L)\right)^{T-1}.$$

The difference of the payoffs of these two allocation rules is

$$\begin{aligned}
&\lambda(L)\delta^{T-1} P(L|L)^{T-1} L - \lambda(L)\delta H P(H|L) \frac{(\delta P(L|L))^{T-1}}{1 - \delta P(L|L)} \\
&- \lambda(H)(H - L)\delta^{T-1} \left(P(H|H) - P(H|L)\right)^{T-1} \\
&= \lambda(L)\delta^{T-1} P(L|L)^{T-1} \big[L - \frac{\delta H P(H|L)}{1 - \delta P(L|L)} \\
&- \frac{\lambda(H)}{\lambda(L)}(H - L) \left(\frac{P(H|H) - P(H|L)}{P(L|L)}\right)^{T-1} \big].
\end{aligned}$$

This difference is positive if

$$L \geq \frac{\delta H P(H|L)}{1 - \delta P(L|L)} + \frac{\lambda(H)}{\lambda(L)}(H - L) \left(\frac{P(H|H) - P(H|L)}{P(L|L)}\right)^{T-1},$$

which holds for sufficiently large T. As a result, the seller will allocate the last good at some stage. Following the same argument as in the proof of Proposition 2, the deadline T_1^* of the last good is the same as that in the case $m = 1$. That is, if $T_1^* > t_2$, then the seller will allocate the last good only when receiving a report "H" along the lowest path before date T_1^*, and allocate the good regardless of the report at date T_1^*. If $T_1^* \leq t_2$, the seller will allocate the good to the buyer right after date t_2.

Suppose that the claim is true for $r-1$, and $T_1^* \geq T_2^* \geq \cdots \geq T_{r-1}^*$. Now we consider the allocation rule of the r-th-to-last good along the lowest path. We take $t_m, \ldots, t_{r+1}$ as given. The aim is to find the deadline to allocate the r-th-to-last good. We assume that $t_{r+1} < T_{r-1}^* < \cdots < T_1^*$ for simplicity.[7] There are three possibilities.

1. If the seller allocates the r-th-to-last good at some stage i with $t_{r+1} < i < T_{r-1}^*$ regardless of the report, then he gets the following expected payoff after stage t_{r+1},

$$
\begin{aligned}
&\lambda(L)\delta P(H|L) \sum_{t_{r+1}-1 \leq k \leq i-2} \delta^k P(L|L)^k \left[f_1(H) + \cdots + f_r(H)\right] \\
&+ \lambda(L)\delta^{i-1} P(L|L)^{i-1} L \\
&+ \lambda(L)\delta P(H|L) \sum_{i-1 \leq k \leq T_{r-1}^*-2} \delta^k P(L|L)^k \left[f_1(H) + \cdots + f_{r-1}(H)\right] \\
&+ \lambda(L)\delta^{T_{r-1}^*-1} P(L|L)^{T_{r-1}^*-1} L \\
&+ \cdots \\
&+ \lambda(L)\delta P(H|L) \sum_{T_2^*-1 \leq k \leq T_1^*-2} \delta^k P(L|L)^k f_1(H) \\
&+ \lambda(L)\delta^{T_1^*-1} P(L|L)^{T_1^*-1} L \\
&- \lambda(H)(H-L)\left[\delta^{i-1}\left(P(H|H) - P(H|L)\right)^{i-1} + \sum_{1 \leq k \leq r-1} \delta^{T_k^*-1}\left(P(H|H) - P(H|L)\right)^{T_k^*-1}\right].
\end{aligned}
$$

The term $\lambda(L)\delta P(H|L)\delta^k P(L|L)^k \left[f_1(H) + \cdots + f_r(H)\right]$ represents the expected payoff in the case that the r-th-to-last good is allocated upon a report "H" before stage i, and then the rest is allocated in the subsequent stages. The term $\lambda(L)\delta^{i-1}P(L|L)^{i-1}L$ is the expected payoff contributed by the r-th-to-last good in the case that the report after stage t_{r+1} is always "L" until stage i. The term $\lambda(L)\delta P(H|L)\delta^k P(L|L)^k \left[f_1(H) + \cdots + f_{r-1}(H)\right]$ describes the payoff in the case that the $r-1$-th-to-last good is allocated upon a report "H" after stage i, but before stage T_{r-1}^*, and then the rest is allocated in the subsequent stages. The term $\lambda(L)\delta^{T_{r-1}^*-1}P(L|L)^{T_{r-1}^*-1}L$ is the expected

[7] Otherwise, the calculation below is the same by taking $T_{r-1}^* = t_{r+1} + 1$, and $T_j^* = T_{j+1}^* + 1$ for $1 \leq j \leq r-2$.

payoff contributed by the $r-1$-th-to-last good in the case that the report is always "L" until stage T^*_{r-1}. The others can be explained similarly. The last term is the rent left to the buyer.

2. Suppose that the seller allocates the r-th-to-last good at stage $i \geq T^*_{r-1}$. Let $T^*_0 = \infty$, and

   $I_j = \{i\colon i \geq T^*_{r-1}, j$ is the largest j' such that $j' \leq r-2, T^*_{j'} + j' > i + r - 1\}$[8]

   for $0 \leq j \leq r-2$. Then $\{I_j\}_{0\leq j\leq r-2}$ is a partition of the sequence $\{T^*_{r-1}, T^*_{r-1}+1, \ldots\}$ such that $i > i'$ for any $i \in I_j$ and $i' \in I_{j'}$ with $j < j'$. If $i \in I_j$, then he gets the following expected payoff after stage t_{r+1},

$$
\begin{aligned}
\Pi^r(j,i) = {} & \lambda(L)\delta P(H|L) \sum_{t_{r+1}-1\leq k\leq i-2} \delta^k P(L|L)^k \left[f_1(H) + \cdots + f_r(H)\right] \\
& + \lambda(L)\delta^{i-1} P(L|L)^{i-1} f_1(L) \\
& + \lambda(L)\delta P(H|L)\delta^{i-1} P(L|L)^{i-1} \left[f_1(H) + \cdots + f_{r-1}(H)\right] \\
& + \lambda(L)\delta^{i} P(L|L)^{i} f_1(L) \\
& + \cdots \\
& + \lambda(L)\delta P(H|L)\delta^{i+r-j-3} P(L|L)^{i+r-j-3} \left[f_1(H) + \cdots + f_{j+1}(H)\right] \\
& + \lambda(L)\delta^{i+r-j-2} P(L|L)^{i+r-j-2} f_1(L) \\
& + \lambda(L)\delta P(H|L) \sum_{i+r-j-2\leq k\leq T^*_j-2} \delta^k P(L|L)^k \left[f_1(H) + \cdots + f_j(H)\right] \\
& + \lambda(L)\delta^{T^*_j-1} P(L|L)^{T^*_j-1} L \\
& + \cdots \\
& + \lambda(L)\delta P(H|L) \sum_{T^*_2-1\leq k\leq T^*_1-2} \delta^k P(L|L)^k f_1(H) \\
& + \lambda(L)\delta^{T^*_1-1} P(L|L)^{T^*_1-1} L \\
& - \lambda(H)(H-L)\Big[\sum_{0\leq k\leq r-j-1} \delta^{i+k-1} \left(P(H|H) - P(H|L)\right)^{i+k-1}
\end{aligned}
$$

[8] If $i \in I_j$, then $i+1 > T^*_{r-1}$ and hence the $r-1$-th-to-last unit will be allocated in stage $i+1$ immediate. Then j is the first number that the j-th-to-last good could be allocated within its deadline. That is, $T^*_j > i + (r-1-j)$, where i is the stage at which the r-th-to-last unit is allocated, and $r-1-j$ is the number of units that are allocated after their deadlines due to the allocation date of the i-th-to-last unit.

$$+\sum_{1\le k\le j} \delta^{T_k^*-1}\left(P(H|H)-P(H|L)\right)^{T_k^*-1}\Big].$$

The term $\lambda(L)\delta P(H|L)\delta^k P(L|L)^k [f_1(H)+\cdots+f_r(H)]$ represents the expected payoff in the case that the r-th-to-last good is allocated upon a report "H" before stage i, and then the rest is allocated in the subsequent stages. The term $\lambda(L)\delta^{i-1}P(L|L)^{i-1}L$ is the expected payoff contributed by the r-th-to-last good in the case that the report after stage t_{r+1} is always "L" until stage i. The term $\lambda(L)\delta P(H|L)\delta^{i-1}P(L|L)^{i-1}[f_1(H)+\cdots+f_{r-1}(H)]$ describes the expected payoff in the case that the r-th-to-last good is allocated until stage i along the lowest path and then the report in the next stage is "H". The other terms can be explained similarly.

3. If the seller never allocates the r-th-to-last good along the lowest path, then he gets the following expected payoff after stage t_{r+1},

$$\lambda(L)\delta P(H|L)\sum_{k\ge t_{r+1}-1}\delta^k P(L|L)^k [f_1(H)+\cdots+f_r(H)].$$

It is obvious that case (3) is the limit of case (2) by taking i to the positive infinity.

We first consider case (1) above. After some simple algebras, the expected payoff is

$$\begin{aligned}
&C_1-\lambda(L)\delta P(H|L)f_r(H)\frac{\delta^{i-1}P(L|L)^{i-1}}{1-\delta P(L|L)}+\lambda(L)\delta^{i-1}P(L|L)^{i-1}L\\
&-\lambda(H)(H-L)\delta^{i-1}\left(P(H|H)-P(H|L)\right)^{i-1}\\
&=C_1+\lambda(L)\delta^{i-1}P(L|L)^{i-1}\big[L-\frac{\delta P(H|L)f_r(H)}{1-\delta P(L|L)}\\
&-\frac{\lambda(H)}{\lambda(L)}(H-L)\left(\frac{P(H|H)-P(H|L)}{P(L|L)}\right)^{i-1}\big],
\end{aligned}$$

where C_1 is some constant independent of the choice of i.[9] Following the same

[9] In particular,

$$C_1=\lambda(L)\delta P(H|L)f_r(H)\frac{\delta^{t_{r+1}-1}P(L|L)^{t_{r+1}-1}}{1-\delta P(L|L)}$$

argument as in the proof of Proposition 2, one can show that the payoff in the above equation is increasing in i up to some point and then decreasing. In particular, the optimal stage $\tilde{t}_r$ to allocate this good along the lowest path is the smallest t such that

$$
\begin{aligned}
L \geq\ & \delta \left(LP(L|L) + f_r(H)P(H|L) \right) \\
& + \frac{\lambda(H)}{\lambda(L)}(H-L)\left[1-\delta(P(H|H)-P(H|L))\right]\left[\frac{P(H|H)-P(H|L)}{P(L|L)}\right]^{t-1}.
\end{aligned}
$$

Because $f_r(H) < f_{r-1}(H)$, $\tilde{t}_r \leq T^*_{r-1}$.

We then consider case (2). Notice that

$$
\begin{aligned}
\Pi^r(j,i) =\ & \Pi^{r-1}(j,i+1) + \lambda(L)\delta P(H|L) \sum_{t_{r+1}-1 \leq k \leq i-2} \delta^k P(L|L)^k f_r(H) \\
& + \lambda(L)\delta^{i-1}P(L|L)^{i-1} f_1(L) \\
& - \lambda(H)(H-L)\delta^{i-1}\left(P(H|H)-P(H|L)\right)^{i-1} \\
& = \Pi^{r-1}(j,i+1) \\
& + \lambda(L)\delta P(H|L) f_r(H) \frac{\delta^{t_{r+1}-1}P(L|L)^{t_{r+1}-1}}{1-\delta P(L|L)} - \lambda(L)\delta P(H|L) f_r(H) \frac{\delta^{i-1}P(L|L)^{i-1}}{1-\delta P(L|L)} \\
& + \lambda(L)\delta^{i-1}P(L|L)^{i-1} f_1(L) \\
& - \lambda(H)(H-L)\delta^{i-1}\left(P(H|H)-P(H|L)\right)^{i-1} \\
& = \Pi^{r-1}(j,i+1) + C \\
& + \lambda(L)\delta^{i-1}P(L|L)^{i-1}\big[f_1(L) - \frac{\delta P(H|L) f_r(H)}{1-\delta P(L|L)}
\end{aligned}
$$

$$
\begin{aligned}
& + \lambda(L)\delta P(H|L) \sum_{t_{r+1}-1 \leq k \leq T^*_{r-1}-2} \delta^k P(L|L)^k \left[f_1(H) + \cdots + f_{r-1}(H)\right] \\
& + \lambda(L)\delta^{T^*_{r-1}-1}P(L|L)^{T^*_{r-1}-1} L \\
& + \cdots \\
& + \lambda(L)\delta P(H|L) \sum_{T^*_2-1 \leq k \leq T^*_1-2} \delta^k P(L|L)^k f_1(H) \\
& + \lambda(L)\delta^{T^*_1-1}P(L|L)^{T^*_1-1} L \\
& - \lambda(H)(H-L) \sum_{1 \leq k \leq r-1} \delta^{T^*_k-1}\left(P(H|H)-P(H|L)\right)^{T^*_k-1},
\end{aligned}
$$

which does not depend on i.

$$
- \frac{\lambda(H)}{\lambda(L)}(H-L)\left(\frac{P(H|H)-P(H|L)}{P(L|L)}\right)^{i-1}],
$$

where $C = \lambda(L)\delta P(H|L) f_r(H) \frac{\delta^{t_{r+1}-1} P(L|L)^{t_{r+1}-1}}{1-\delta P(L|L)}$, and $\Pi^{r-1}(j, i+1)$ is the payoff in the case that there are only $r-1$ goods in stock and the seller allocates the $r-1$-th-to-last good at stage $i+1$. Due to our induction hypothesis and the fact that $i+1 > T^*_{r-1}$, $\Pi^{r-1}(j, i+1)$ must be (weakly) less than the payoff by setting the deadline at the date T^*_{r-1}. In addition, by the argument for the case (1) above,

$$
\lambda(L)\delta^{i-1}P(L|L)^{i-1}\big[f_1(L) - \frac{\delta P(H|L) f_r(H)}{1-\delta P(L|L)} - \frac{\lambda(H)}{\lambda(L)}(H-L)\left(\frac{P(H|H)-P(H|L)}{P(L|L)}\right)^{i-1}]
$$

is increasing in i up to $\tilde{t}_r \leq T^*_{r-1}$ and then decreasing.

As a result, the seller's expected payoff in case (2) is less than that in case (1), and the seller will allocate the r-th-to-last good at stage $\tilde{t}_r$.[10] That is, $T^*_r = \tilde{t}_r$ is the smallest t such that

$$
\begin{aligned}
L \geq\ & \delta\left(LP(L|L) + f_r(H)P(H|L)\right) \\
& + \frac{\lambda(H)}{\lambda(L)}(H-L)\left[1 - \delta(P(H|H) - P(H|L))\right]\left[\frac{P(H|H)-P(H|L)}{P(L|L)}\right]^{t-1}.
\end{aligned}
$$

This proves our claim. □

[10] Since the payoff in case (3) is the limit of the payoffs in case (2), the seller will not choose the allocation rule in case (3).

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